



# A New 3-D Multistable Chaotic System with Line Equilibrium: Dynamic Analysis and Synchronization

Muhamad Deni Johansyah

*Department of Mathematics, Faculty of Mathematics and Natural Sciences,  
Universitas Padjajaran, Jl. Raya Bandung-Sumedang Km 21, Jatinangor 45363 Indonesia*

*\*Corresponding author email: muhamad.deni@unpad.ac.id*

---

## Abstract

This work introduces a new 3-D chaotic system with a line of equilibrium points. We carry out a detailed dynamic analysis of the proposed chaotic system with five nonlinear terms. We show that the chaotic system exhibits multistability with two coexisting chaotic attractors. We apply integral sliding mode control for the complete synchronization of the new chaotic system with itself as leader-follower systems.

*Keywords:* Chaos, chaotic systems, synchronization, line equilibrium

---

## 1. Introduction

Chaotic systems are nonlinear dynamical systems with a positive Lyapunov exponent and extreme sensitivity to even small changes in their initial states (Qi et al., 2005). Chaos theory has several applications like jerk systems (Sambas et al., 2021a; Qin et al., 2021), mechanical oscillators (Belato et al., 2021), neurons (Baysal et al., 2021), fuzzy systems (Sukono et al., 2020; Xia et al., 2020), circuits (Sambas et al., 2021b; Vaidyanathan et al., 2021; Abro and Atangana, 2021), neural networks (Gao, et al., 2021), secure communications (Zhou, et al., 2021; Pan, et al., 2021), image encryption (Sambas et al., 2020; Ouannas, et al., 2021), finance (Bambe, et al., 2020; Moutsinga et al., 2020), robotic (Vaidyanathan et al., 2017) etc.

In the chaos literature, there is good interest shown in finding of chaotic systems with line equilibrium points (Jalal, et al., 2020; Sambas et al., 2019). Such systems are said to possess hidden attractors as they possess an infinite number of equilibrium points (Tlelo-Cuautle et al., 2017). In this research paper, we propose a new chaotic system with line equilibrium.

Multistability is a special property of nonlinear dynamics systems which is the coexistence of periodic orbits and/or chaotic attractors for same parameter set but different initial conditions (Chakraborty and Poria, 2019; Mobayen et al., 2021). In this work, we show that the new chaotic system has multistability with coexisting attractors.

Control of dynamical systems exhibiting chaos is an active research area in the control literature (Peng and Chen, 2008). Many control methods are used in control engineering for the control and synchronization of chaotic systems such as nonlinear control (Cai and Tan, 2007), adaptive control (Vaidyanathan, 2015), backstepping control (Yassen, 2006), sliding mode control (Jang et al., 2002), etc. In this work, we use integral sliding mode control to derive global synchronization of the new chaotic systems taken as leader-follower systems with unknown constants. Sliding mode control has attractive properties such as fast convergence, robustness etc. (Vaidyanathan et al., 2019).

This research work is organized in the following manner. Section 2 gives the mathematical model of the new chaotic system with face-like equilibrium curve. Section 3 investigates the global self-synchronization of the new chaotic systems considered as leader-follower systems using adaptive control. Section 4 contains the conclusions.

## 2. A New Chaotic System with a Line of Equilibrium Points

In this work, we consider a new 3-D system having the dynamics

$$\begin{cases} \dot{y}_1 = y_2 y_3 \\ \dot{y}_2 = y_1 - y_2 \\ \dot{y}_3 = a |y_1| - b y_1^2 - c y_1^4 - d y_2^2 \end{cases} \quad (1)$$

In (1),  $Y = (y_1, y_2, y_3)$  is the state vector and  $(a, b, c, d)$  is the parameter vector.

We show that the system (1) exhibits a *chaotic attractor* when the parameter vector is taken as  $a = 6, b = 0.5, c = 2, d = 0.1$  (2)

For MATLAB plot, we take the initial state of the chaotic system (1) as

$$y_1(0) = 0.4, \quad y_2(0) = 0.2, \quad y_3(0) = 0.4 \quad (3)$$

Using Wolf algorithm (Wolf, et al., 1985), we calculate the Lyapunov characteristic exponents (LCE) in MATLAB for the 3-D system (1) for the parameters (2) and the initial state (3) for  $T = 1E5$  seconds as follows:

$$\mu_1 = 0.1812, \quad \mu_2 = 0, \quad \mu_3 = -1.1782 \quad (4)$$

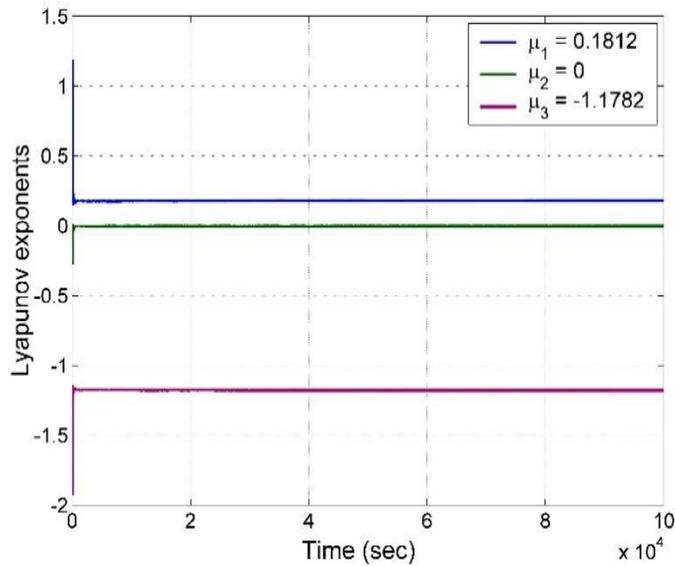
Figure 1 shows the Lyapunov exponents of the new chaotic system (1) for  $(a, b, c, d) = (6, 0.5, 2, 0.1)$  and the initial state  $Y(0) = (0.4, 0.2, 0.4)$ .

The system (1) is chaotic since it possesses a positive Lyapunov characteristic exponent  $\mu_1 = 0.1812$ . We also find that  $\mu_1 + \mu_2 + \mu_3 = -0.9970 < 0$ . Since the sum of the Lyapunov characteristic exponents is negative, we deduce that the 3-D system (1) is dissipative.

The Kaplan-Yorke dimension of the new chaotic system (1) is calculated as follows:

$$D_{KY} = 2 + \frac{\mu_1 + \mu_2}{|\mu_3|} = 2 + \frac{0.1812 + 0}{1.1782} = 2.1538 \quad (5)$$

Figure 2 shows the MATLAB plots of the new chaotic system (1) in various coordinate planes and the 3-D space for the parameter vector  $(a, b, c, d) = (6, 0.5, 2, 0.1)$  and  $Y(0) = (0.4, 0.2, 0.4)$ .



**Figure 1.** Lyapunov exponents of the new 3-D chaotic system (1) for the parameter state  $(a, b, c, d) = (6, 0.5, 2, 0.1)$  and initial state  $Y(0) = (0.4, 0.2, 0.4)$ .

The equilibrium points of the chaotic system (1) are obtained by solving the system of equations:

$$y_2 y_3 = 0 \tag{6a}$$

$$y_1 - y_2 = 0 \tag{6b}$$

$$a | y_1 | - b y_1^2 - c y_1^4 - d y_2^2 = 0 \tag{6c}$$

From (6b), we see that  $y_1 = y_2$ . Hence, the equations (6a) reduce to the system:

$$y_1 y_3 = 0 \tag{7a}$$

$$a | y_1 | - (b + d) y_1^2 - c y_1^4 = 0 \tag{7b}$$

From (7a),  $y_1 = 0$  or  $y_3 = 0$ .

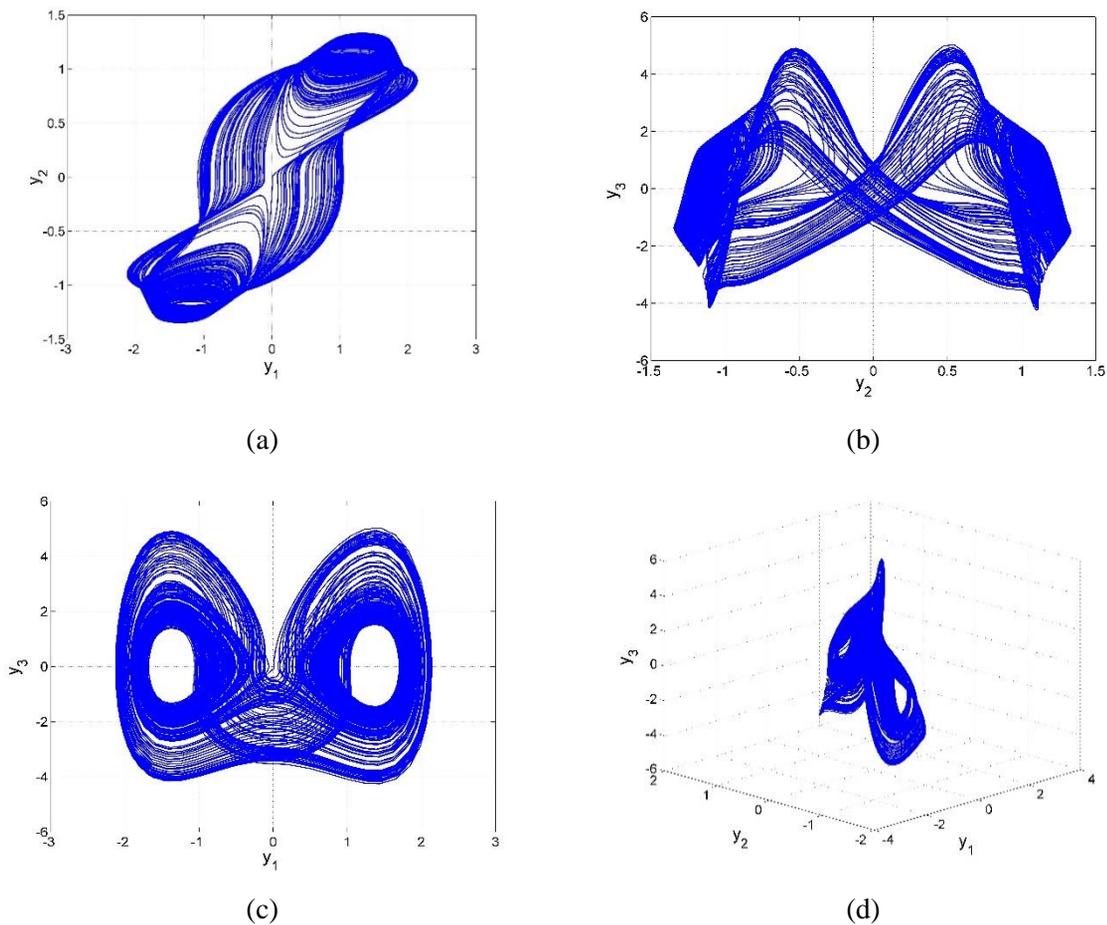
If  $y_1 = 0$ , then  $y_2 = y_1 = 0$ . In this case, the  $y_3$  - axis is a line equilibrium for the system (1).

If  $y_1 \neq 0$ , then  $y_3 = 0$ . Solving (7b) for the parameter values  $(a, b, c, d) = (6, 0.5, 2, 0.1)$ , we get three roots namely  $y_1 = 0$ ,  $y_1 = 1.3730$  and  $y_1 = -1.3730$ .

Since  $y_2 = y_1$ , we get corresponding values of  $y_2$ , viz.  $y_2 = 0$ ,  $y_2 = 1.3730$  and  $y_2 = -1.3730$ .

Thus, we get three equilibrium points on the  $(y_1, y_2)$ - plane as follows:  $\Omega_0 = (0, 0, 0)$ ,  $\Omega_1 = (1.3730, 1.3730, 0)$  and  $\Omega_2 = (-1.3730, -1.3730, 0)$ . The point  $\Omega_0$  lies on the  $y_3$ - axis. Thus, this is already included in the equilibrium points of the system (1).

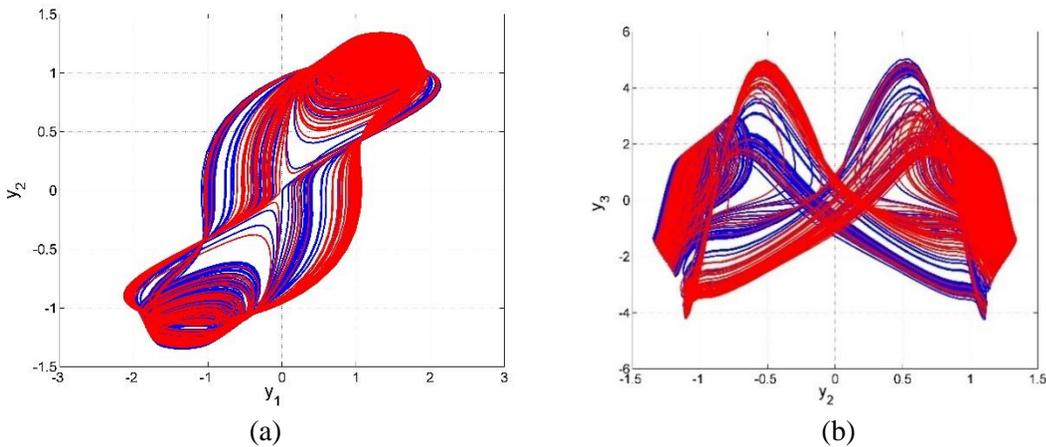
Hence, the equilibrium points of the system (1) consists of the  $y_3$ - axis and the two equilibria on the  $(y_1, y_2)$ - plane given by  $\Omega_1 = (1.3730, 1.3730, 0)$  and  $\Omega_2 = (-1.3730, -1.3730, 0)$ . It is easy to verify that  $\Omega_1$  and  $\Omega_2$  are unstable saddle focus points for the chaotic system (1).



**Figure 2.** MATLAB signal plots of the new 3-D chaotic system (1) for  $(a, b, c, d) = (6, 0.5, 2, 0.1)$  and  $Y(0) = (0.4, 0.2, 0.4)$ .

Multistability, namely coexisting attractors with same parameters but different initial values, is an interesting nonlinear phenomenon in chaotic systems.

When fixing the parameters as  $(a, b, c, d) = (6, 0.5, 2, 0.1)$  and the initial conditions as  $Y_0 = (0.4, 0.2, 0.4)$ , (blue),  $Z_0 = (-0.8, -0.8, -0.8)$  (red), two coexisting chaotic attractors are obtained for the chaotic system (1) as shown in Figure 3.



**Figure 3.** Phase portraits of the coexisting chaotic attractors of the 3-D system (1) for  $(a, b, c, d) = (6, 0.5, 2, 0.1)$  : (a)  $(y_1, y_2)$  – plane and (b)  $(y_2, y_3)$  – plane.

### 3. Global Synchronization of the New Chaotic Systems with Line Equilibrium via Integral Sliding Mode Control

As a control application, we employ integral sliding mode control for the global synchronization between the states of the new chaotic systems taken as *leader-follower* systems.

As the leader system, we consider the new chaotic system with line equilibrium described by

$$\begin{cases} \dot{y}_1 = y_2 y_3 \\ \dot{y}_2 = y_1 - y_2 \\ \dot{y}_3 = a |y_1| - b y_1^2 - c y_1^4 - d y_2^2 \end{cases} \quad (8)$$

We denote the state of the leader system (9) as  $Y = (y_1, y_2, y_3)$ .

As the follower system, we take the controlled chaotic system with line equilibrium described by

$$\begin{cases} \dot{z}_1 = z_2 z_3 + v_1 \\ \dot{z}_2 = z_1 - z_2 + v_2 \\ \dot{z}_3 = a |z_1| - b z_1^2 - c z_1^4 - d z_2^2 + v_3 \end{cases} \quad (9)$$

We denote the state of the follower system (10) as  $Z = (z_1, z_2, z_3)$ .

In the system (9),  $v = (v_1, v_2, v_3)$  is an integral sliding mode control to be designed using sliding mode control theory.

The synchronization errors between the chaotic systems (8) and (9) are defined in the following manner:

$$\begin{cases} \varepsilon_1 = z_1 - y_1 \\ \varepsilon_2 = z_2 - y_2 \\ \varepsilon_3 = z_3 - y_3 \end{cases} \quad (10)$$

We obtain the following system for the error dynamics:

$$\begin{cases} \dot{\varepsilon}_1 = z_2 z_3 - y_2 y_3 + v_1 \\ \dot{\varepsilon}_2 = \varepsilon_1 - \varepsilon_2 + v_2 \\ \dot{\varepsilon}_3 = a(|z_1| - |y_1|) - b(z_1^2 - y_1^2) - c(z_1^4 - y_1^4) - d(z_2^2 - y_2^2) + v_3 \end{cases} \quad (11)$$

In the ISMC design, an integral sliding manifold is defined for each error variable as follows:

$$\begin{cases} S_1 = \varepsilon_1 + \alpha_1 \int_0^t \varepsilon_1(\theta) d\theta \\ S_2 = \varepsilon_2 + \alpha_2 \int_0^t \varepsilon_2(\theta) d\theta \\ S_3 = \varepsilon_3 + \alpha_3 \int_0^t \varepsilon_3(\theta) d\theta \end{cases} \quad (12)$$

From (12), we deduce the following:

$$\begin{cases} \dot{S}_1 = \dot{\varepsilon}_1 + \alpha_1 \varepsilon_1 \\ \dot{S}_2 = \dot{\varepsilon}_2 + \alpha_2 \varepsilon_2 \\ \dot{S}_3 = \dot{\varepsilon}_3 + \alpha_3 \varepsilon_3 \end{cases} \quad (13)$$

In the ISMC design, we assume that  $\alpha_i > 0$  for  $i = 1, 2, 3$ .

Based on the exponential reaching law [48], we set the following:

$$\begin{cases} \dot{S}_1 = -\beta_1 \operatorname{sgn}(S_1) - K_1 S_1 \\ \dot{S}_2 = -\beta_2 \operatorname{sgn}(S_2) - K_2 S_2 \\ \dot{S}_3 = -\beta_3 \operatorname{sgn}(S_3) - K_3 S_3 \end{cases} \quad (14)$$

By comparing the equations (13) and (14), we get the following:

$$\begin{cases} \dot{\varepsilon}_1 + \alpha_1 \varepsilon_1 = -\beta_1 \operatorname{sgn}(S_1) - K_1 S_1 \\ \dot{\varepsilon}_2 + \alpha_2 \varepsilon_2 = -\beta_2 \operatorname{sgn}(S_2) - K_2 S_2 \\ \dot{\varepsilon}_3 + \alpha_3 \varepsilon_3 = -\beta_3 \operatorname{sgn}(S_3) - K_3 S_3 \end{cases} \quad (15)$$

We combine the equations (11) and (15) to obtain the following:

$$\begin{cases} z_2 z_3 - y_2 y_3 + v_1 + \alpha_1 \varepsilon_1 = -\beta_1 \operatorname{sgn}(S_1) - K_1 S_1 \\ \varepsilon_1 - \varepsilon_2 + v_2 + \alpha_2 \varepsilon_2 = -\beta_2 \operatorname{sgn}(S_2) - K_2 S_2 \\ a(|z_1| - |y_1|) - b(z_1^2 - y_1^2) - c(z_1^4 - y_1^4) - d(z_2^2 - y_2^2) + v_3 + \alpha_3 \varepsilon_3 = -\beta_3 \operatorname{sgn}(S_3) - K_3 S_3 \end{cases} \quad (16)$$

From Eq. (16), we obtain the required sliding mode control law as follows:

$$\begin{cases} v_1 = -z_2 z_3 + y_2 y_3 - \alpha_1 \varepsilon_1 - \beta_1 \operatorname{sgn}(S_1) - K_1 S_1 \\ v_2 = -\varepsilon_1 + \varepsilon_2 - \alpha_2 \varepsilon_2 - \beta_2 \operatorname{sgn}(S_2) - K_2 S_2 \\ v_3 = -a(|z_1| - |y_1|) + b(z_1^2 - y_1^2) + c(z_1^4 - y_1^4) + d(z_2^2 - y_2^2) - \alpha_3 \varepsilon_3 - \beta_3 \operatorname{sgn}(S_3) - K_3 S_3 \end{cases} \quad (17)$$

**Theorem 1.** The new chaotic systems (8) and (9) with line equilibrium points are globally and asymptotically synchronized for all initial conditions  $Y(0), Z(0) \in R^3$  by the integral sliding mode controller (17), where the constants  $\alpha_i, \beta_i, K_i, (i = 1, 2, 3)$  are all positive.

*Proof.* We establish this theorem using Lyapunov stability theory (Khalil, 2001).

First, we consider the quadratic and positive definite Lyapunov function defined by

$$V(S_1, S_2, S_3) = \frac{1}{2} (S_1^2 + S_2^2 + S_3^2) \quad (18)$$

We determine the time-derivative of  $V$  as follows:

$$\dot{V} = \sum_{i=1}^3 S_i [-\beta_i \operatorname{sgn}(S_i) - K_i S_i] = \sum_{i=1}^3 [-\beta_i |S_i| - K_i S_i^2] \quad (19)$$

From (19), we see that  $\dot{V}$  is negative definite at all points of  $R^3$ . Using Lyapunov stability theory (Khalil, 2001), we conclude that  $S_i(t) \rightarrow 0$  as  $t \rightarrow \infty$  for each  $i = 1, 2, 3$ .

Hence, it follows that  $\varepsilon_i(t) \rightarrow 0$  as  $t \rightarrow \infty$  for each  $i = 1, 2, 3$ . This completes the proof. ■

For MATLAB simulations, we take  $(a, b, c, d) = (6, 0.5, 2, 0.1)$ .

We take the sliding constants as follows:  $\alpha_1 = \alpha_2 = \alpha_3 = 0.2$  and  $\beta_1 = \beta_2 = \beta_3 = 0.2$ .

We take the gain constants as  $K_i = 20$  for each  $i = 1, 2, 3$ .

We take the initial state of the new chaotic system (8) as

$$y_1(0) = 0.7, \quad y_2(0) = 3.4, \quad y_3(0) = 1.5 \quad (20)$$

We also consider the initial state of the new chaotic system (9) as

$$z_1(0) = 2.1, \quad z_2(0) = 1.6, \quad z_3(0) = 4.8 \quad (21)$$

Figures 5-8 shows the complete synchronization of the new chaotic systems (8) and (9).

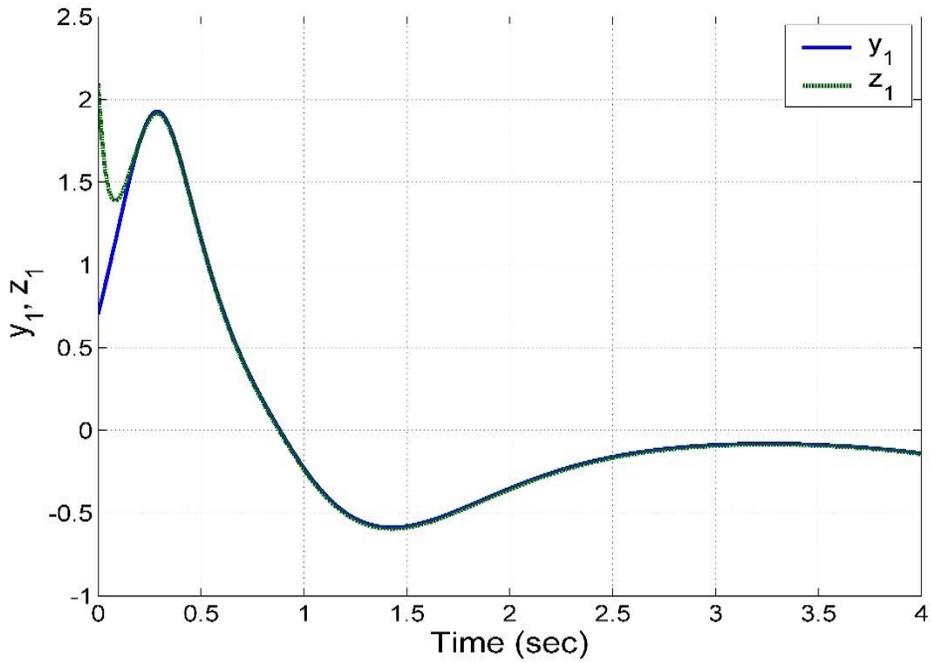
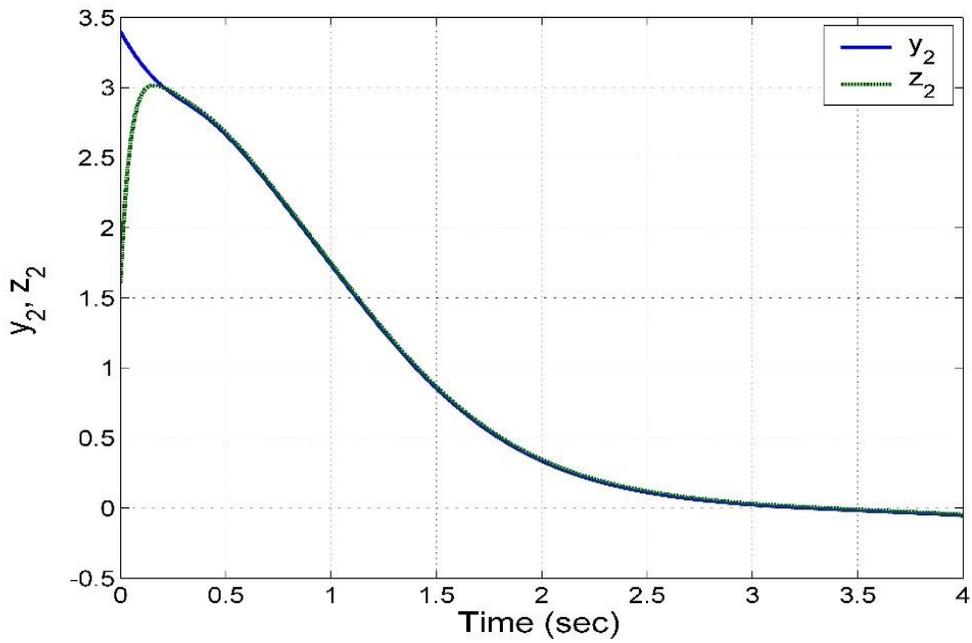
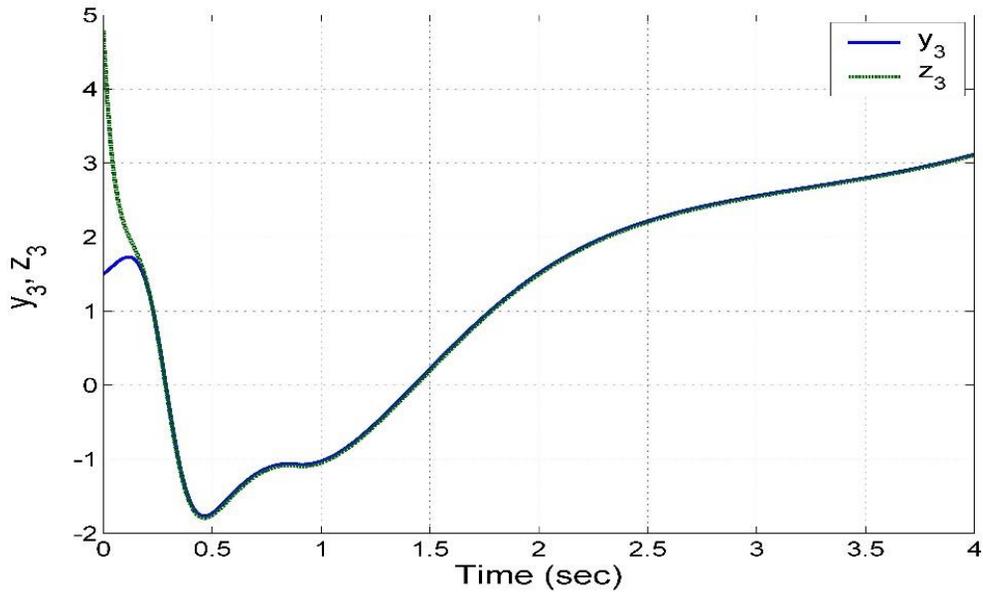


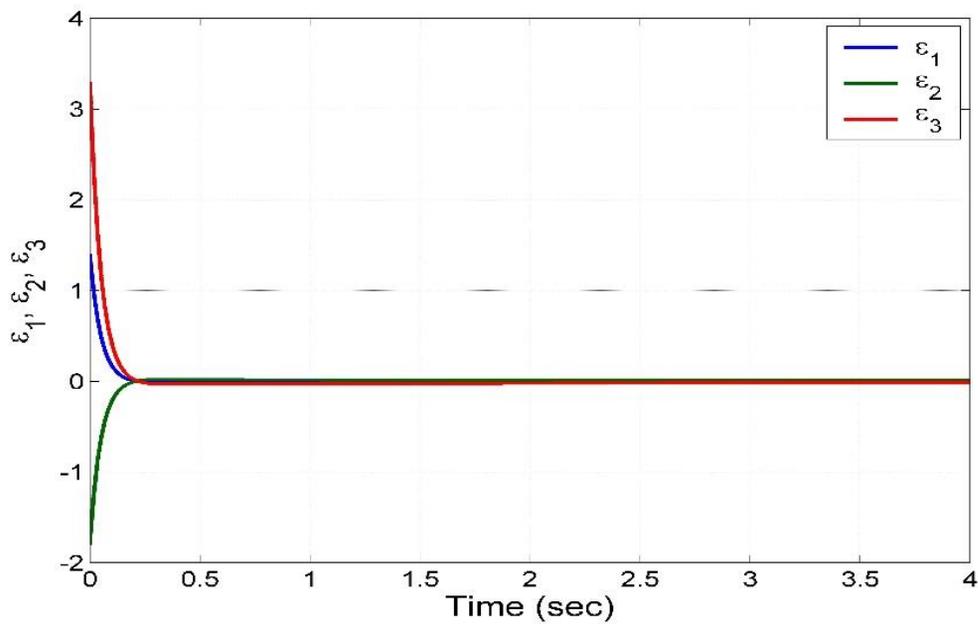
Figure 5. Complete synchronization of the states  $y_1, z_1$  of the chaotic systems (8) and (9)



**Figure 6.** Complete synchronization of the states  $y_2, z_2$  of the chaotic systems (8) and (9)



**Figure 7.** Complete synchronization of the states  $y_3, z_3$  of the chaotic systems (8) and (9)



**Figure 8.** Time history of the synchronization errors between the chaotic systems (8) and (9)

## 4. Conclusion

In this work, we briefed on a new 3-D chaotic system with a line of equilibrium points. We presented a dynamic analysis of the proposed chaotic system with five nonlinear terms such as Lyapunov exponents, Kaplan-Yorke dimension, etc. We exhibited that the new chaotic system with line equilibrium has the special property of multistability with two coexisting chaotic attractors. Using integral sliding mode control, we derived new control results for the complete synchronization of the new chaotic system with itself as leader-follower systems.

## References

- Abro, K. A. and Atangana, A. (2021). Numerical Study and Chaotic Analysis of Meminductor and Memcapacitor Through Fractal–Fractional Differential Operator. *Arabian Journal for Science and Engineering*, 46, 857-871.
- Baysal, V., Erkan, E., & Yilmaz, E. (2021). Impacts of autapse on chaotic resonance in single neurons and small-world neuronal networks. *Philosophical Transactions of the Royal Society A*, 379(2198), 20200237.
- Belato, D., Weber, H. I., Balthazar, J. M., and Mook, D. T. (2001). Chaotic vibrations of a nonideal electro-mechanical system. *International Journal of Solids and Structures*, 38(10-13), 1699-1706.
- Cai, G., and Tan, Z. (2007). Chaos synchronization of a new chaotic system via nonlinear control. *Journal of Uncertain systems*, 1(3), 235-240.
- Chakraborty, P., and Poria, S. (2019). Extreme multistable synchronisation in coupled dynamical systems. *Pramana*, 93(2), 1-13.
- Gao, S., Zhang, Y., Zhang, Y., and Zhang, G. (2020). Elman Neural Network Soft-Sensor Model of PVC Polymerization Process Optimized by Chaos Beetle Antennae Search Algorithm. *IEEE Sensors Journal*, 21(3), 3544-3551.
- Jalal, A. A., Amen, A. I., and Sulaiman, N. A. (2020). Darboux integrability of the simple chaotic flow with a line equilibria differential system. *Chaos, Solitons & Fractals*, 135, 109712.
- Jang, M. J., Chen, C. L., & Chen, C. O. K. (2002). Sliding mode control of chaos in the cubic Chua's circuit system. *International Journal of Bifurcation and Chaos*, 12(06), 1437-1449.
- Mobayen, S., Fekih, A., Vaidyanathan, S., & Sambas, A. (2021). Chameleon Chaotic Systems With Quadratic Nonlinearities: An Adaptive Finite-Time Sliding Mode Control Approach and Circuit Simulation. *IEEE Access*, 9, 64558-64573.
- Moutsinga, C. R. B., Pindza, E., and Mare, E. (2020). A robust spectral integral method for solving chaotic finance systems. *Alexandria Engineering Journal*, 59(2), 601-611.

- Ouannas, A., Karouma, A., Grassi, G., and Pham, V. T. (2021). A novel secure communications scheme based on chaotic modulation, recursive encryption and chaotic masking. *Alexandria Engineering Journal*, 60(1), 1873-1884.
- Pan, J., Ding, Q., and Du, B. (2012). A new improved scheme of chaotic masking secure communication based on Lorenz system. *International Journal of Bifurcation and Chaos*, 22(05), 1250125.
- Peng, C. C., and Chen, C. L. (2008). Robust chaotic control of Lorenz system by backstepping design. *Chaos, Solitons & Fractals*, 37(2), 598-608.
- Sambas, A., Vaidyanathan, S., Zhang, S., Zeng, Y., Mohamed, M. A., and Mamat, M. (2019). A new double-wing chaotic system with coexisting attractors and line equilibrium: bifurcation analysis and electronic circuit simulation. *IEEE Access*, 7, 115454-115462.
- Sambas, A., Vaidyanathan, S., Tlelo-Cuautle, E., Abd-El-Atty, B., Abd El-Latif, A. A., Guillén-Fernández, O., Hidayat, Y and Gundara, G. (2020). A 3-D multi-stable system with a peanut-shaped equilibrium curve: Circuit design, FPGA realization, and an application to image encryption. *IEEE Access*, 8, 137116-137132.
- Sambas, A., Vaidyanathan, S., Zhang, S., Mohamed, M. A., Zeng, Y., and Azar, A. T. (2021a). A new 3-D chaotic jerk system with a saddle-focus rest point at the origin, its active backstepping control, and circuit realization. In *Backstepping Control of Nonlinear Dynamical Systems* (pp. 95-114). Academic Press.
- Sambas, A., Vaidyanathan, S., Bonny, T., Zhang, S., Hidayat, Y., Gundara, G., and Mamat, M. (2021b). Mathematical Model and FPGA Realization of a Multi-Stable Chaotic Dynamical System with a Closed Butterfly-Like Curve of Equilibrium Points. *Applied Sciences*, 11(2), 788.
- Sukono, Sambas, A., He, S., Liu, H., Vaidyanathan, S., Hidayat, Y., and Saputra, J. (2020). Dynamical analysis and adaptive fuzzy control for the fractional-order financial risk chaotic system. *Advances in Difference Equations*, 674(1), 1-12.
- Qi, G., Chen, G., Du, S., Chen, Z., and Yuan, Z. (2005). Analysis of a new chaotic system. *Physica A: Statistical Mechanics and its Applications*, 352(2-4), 295-308.
- Qin, C., Sun, K., and He, S. (2021). Characteristic Analysis of Fractional-Order Memristor-Based Hypogenetic Jerk System and Its DSP Implementation. *Electronics*, 10(7), 841.
- Tlelo-Cuautle, E., de la Fraga, L. G., Pham, V. T., Volos, C., Jafari, S., and de Jesus Quintas-Valles, A. (2017). Dynamics, FPGA realization and application of a chaotic system with an infinite number of equilibrium points. *Nonlinear Dynamics*, 89(2), 1129-1139.
- Vaidyanathan, S. (2015). Adaptive control of the FitzHugh-Nagumo chaotic neuron model. *International Journal of PharmTech Research*, 8(6), 117-127.
- Vaidyanathan, S., Sambas, A., Mamat, M., and Sanjaya, W. S. M. (2017). A new three-dimensional chaotic system with a hidden attractor, circuit design and application in wireless mobile robot. *Archives of Control Sciences*, 27(4), 541-554.

- Vaidyanathan, S., Dolvis, L. G., Jacques, K., Lien, C. H., and Sambas, A. (2019). A new five-dimensional four-wing hyperchaotic system with hidden attractor, its electronic circuit realisation and synchronisation via integral sliding mode control. *International Journal of Modelling, Identification and Control*, 32(1), 30-45.
- Vaidyanathan, S., Tlelo-Cuautle, E., Anand, P. G., Sambas, A., Guillén-Fernández, O., & Zhang, S. (2021). A new conservative chaotic dynamical system with lemniscate equilibrium, its circuit model and FPGA implementation. *International Journal of Automation and Control*, 15(2), 128-148.
- Wolf, A., Swift, J. B., Swinney, H. L., and Vastano, J. A. (1985). Determining Lyapunov exponents from a time series. *Physica D: Nonlinear Phenomena*, 16(3), 285-317.
- Xia, Y., Wang, J., Meng, B., and Chen, X. (2020). Further results on fuzzy sampled-data stabilization of chaotic nonlinear systems. *Applied Mathematics and Computation*, 379, 125225.
- Yassen, M. T. (2006). Chaos control of chaotic dynamical systems using backstepping design. *Chaos, Solitons & Fractals*, 27(2), 537-548.
- Zhou, L., Tan, F., Li, X., & Zhou, L. (2021). A fixed-time synchronization-based secure communication scheme for two-layer hybrid coupled networks. *Neurocomputing*, 433, 131-141.