Available online at https://journal.rescollacomm.com/index.php/ijbesd/index

BES

International Journal of Business, Economics and Social Development

e-ISSN 2722-1156
p-ISSN 27722-1164

Vol. 1, No. 1, pp. 46-54, 2020

Abnormal Portfolio Asset Allocation Model: Review

Nurfadhlina bt Abdul Halim[a,*], Dwi Susanti[b], Alit Kartiwa[b], Endang Soeryana Hasbullah[b]

[b]*Faculty of Science and Technology, Universiti Sains Islam Malaysia, MALAYSIA*
[b]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Indonesia*

**Corresponding author e-mail address: nurfadhlina@usim.edu.my*

Abstract

It has been widely studied how investors will allocate their assets to an investment when the return of assets is normally distributed. In this context usually, the problem of portfolio optimization is analyzed using mean-variance. When asset returns are not normally distributed, the mean-variance analysis may not be appropriate for selecting the optimum portfolio. This paper will examine the consequences of abnormalities in the process of allocating investment portfolio assets. Here will be shown how to adjust the mean-variance standard as a basic framework for asset allocation in cases where asset returns are not normally distributed. We will also discuss the application of the optimum strategies for this problem. Based on the results of literature studies, it can be concluded that the expected utility approximation involves averages, variances, skewness, and kurtosis, and can be extended to even higher moments.

Keywords: Portfolio, expected utility, preference, moment method, Taylor's series.

1. Introduction

How investors will allocate capital when income (return) is not normally distributed. Two approaches can be used to study this problem. The first approach is based on the direct maximization of utility expectations, under alternative assumptions of distribution for capital income (assets). The advantage of this approach is that it provides a real evaluation of the utility of expectations, also that the optimal portfolio is the actual resolution of the original problem (Ruppert, 2004). Furthermore that for most applications, numerical integration is difficult to use to maximize the expectation of the established utility. Consequently, most studies focus on a very small amount of income (two or three) to reduce computational burden (Denuit et al., 2005; Sukono et al., 2019).

The second approach is based on the optimization of a problem that does not require numerical integration. Typically, this approximation includes the moment of income-portfolio distribution. The main difficulty with this approach is how to define high moments that influence the utility of hope. Although various solutions have been introduced, the discussion in this paper focuses on Taylor's expansion of the utility function, which naturally produces utility expectations that depend on the high moment linearity of portfolio income (Batuparan, 2001).

2. Direct Maximum Utility Hope

In the direct maximization of this utility, it will cover optimization issues, case-variance cases, and numerical integration.

2.1. Optimization Problem

Suppose an investor allocates portfolio assets by maximizing the utility of hope for his assets in the next period W_{t+1}. For example, there are n investor wealth that can be bought and sold, with income vectors $r_{t+1} = (r_{1,t+1},...,r_{n,t+1})'$. It is assumed that there are no costs for short-selling. Wealth at the beginning of the period is denoted by W_t, assign the changes equal to one. The next period's wealth is given by $W_{t+1} = (\alpha'_t\, r_{t+1})W_t$, where is the fractional vector of wealth allocated to capital, with the limitation that the weight of the portfolio, at the time t, is one, that is $\alpha'_t\, e = 1$, where e is a vector $(n,1)$. Therefore, income on the portfolio is provided by (Jondeau et al., 2007; Denuit et al., 2005).

$$r_{p,t+1}(\alpha_t) = \alpha'_t\, r_{t+1}.$$

It is assumed that the investor has a utility function U that depends on the level of wealth in the coming period W_{t+1}. Formally, optimal portfolio weights are obtained by maximizing the degree of utility utility

$$\alpha_t^* = \arg\max_{\{\alpha\}} E[U(\alpha'_t\, r_{t+1})] \text{ dengan } \alpha'_t\, e = 1 \tag{1}$$

The first-degree condition of the optimization problem is

$$\frac{\partial E(U(W_{t+1}))}{\partial \alpha_t} = E[U^{(1)}(W_{t+1}) r_{t+1}] = 0\,,$$

where $U^{(j)}$ states the derivative jth of the utility function. Because the weight of the portfolio must amount to one, the first-degree requirement for investor problems reduces to the limit $(n-1)$

$$E\left[U^{(1)}(W_{t+1})\begin{pmatrix} r_{1,t+1} - r_{n,t+1} \\ \cdots \\ r_{n-1,t+1} - r_{n,t+1} \end{pmatrix}\right] = 0$$

alternatively,

$$E[U^{(1)}(W_{t+1})\lambda_{t+1}] = 0 \tag{2}$$

where λ_{t+1} is the income vector from asset 1 to $n-1$ the excess asset n. In some cases, the optimization-level requirement (2) can help get a solution to the optimization problem. But in general, it is more appropriate to directly maximize the utility of hope, write back as

$$E[U(W_{t+1})] = \int \ldots \int U(W_{t+1}) f(r_{t+1}) dr_{1,t+1} \ldots dr_{n,t+1} \tag{3}$$

where $f(r_{t+1})$ stated pdf combined income vectors in time (Beronilla et al., 2007).

2.2. Case Variance Mean

In some cases, the problem of double integration reduces average problems - mere variance. Consider for this case where the utility function is chosen exponential utility and where income is assumed to be normally distributed. The exponential utility function (or HOW, short for Constant Absolute Risk Version) is defined as

$$U(W_{t+1}) = -\exp(-\lambda W_{t+1}) \tag{4}$$

where $\lambda \geq 0$ is the absolute risk aversion coefficient. After all, the expectation income vector $(n,1)$, and the covariance matrix (n,n) for risk assets are successively expressed with $\mu = (\mu_{1,t+1}, \ldots, \mu_{n,t+1})'$ and Σ_{t+1}. Furthermore, if the random variable X is normally distributed, then $E[\exp(X)] = \exp(a + \frac{1}{2} b^2)$. Therefore, problem (1) can be rewritten as

$$E[-\exp(-\lambda W_{t+1})] = -E[\exp(-\lambda(\alpha'_t \ r_{t+1}))] = -\exp\left(-\lambda \mu_{p,t+1} + \frac{1}{2} \lambda^2 \sigma^2_{p,t+1}\right)$$

where $\mu_{p,t+1} = \alpha'_t \ \mu_{t+1}$ is the expected gross income of the portfolio, and $\sigma^2_{p,t+1} = \alpha'_t \ \Sigma_{t+1} \alpha_t$ is the variance of portfolio income. So maximizing $E[U(W_{t+1})]$ is equivalent to maximizing $(\mu_{p,t+1} + \frac{1}{2} \lambda \sigma^2_{p,t+1})$ form, which is the average objective function - variance. On the other hand, some utility functions state directly the average size - the variance for variable income distribution. This is the case of the quadratic utility function: $U(W) = a_0 + a_1 W + a_2 W^2$. The reason is simple that the expected utility function $E[U(W)]$ only includes the mean and variance of the distribution. Therefore, if (i) the income distribution for a portfolio is asymmetric, (ii) the investor's utility function is a degree higher than quadratic, and (iii) the average and variance are not completely determined by the distribution, then the third or higher moment and the coefficient mark must be considered (Denuit et al., 2005).

2.3. Numerical Integration

Tauchen and Hussey (1991) provide a numerical solution to equations such as (2.3) with quadratic. A quadratic rule for functions $h(u)$ where $u \in R^n$ and pdf $f(u)$ are a set of points $\{u_i\}$, $i = 1, \ldots, M$, and correspond with weights $\{w_i\}$ such that

$$\int_a^b h(u) f(u) du = \sum_{i=1}^{M} w_i h(u_i)$$

The choice of abscissa u_i and weight w_i, $i=1,...,n$ depends only on the pdf f, but not on the functions h that are integrated. For a univariate quadratic, the Gauss rule is a discrete approximation to be f determined by the moment method using the moment beyond the above $2M-1$.

The multiple-dimension quadrature is more in demand, because it has to calculate the integral dimension- n

$$\int_a^b h(u)f(u)du = \sum_{i_1=1}^{M}...\sum_{i_n=1}^{M} w_{i_1}^1...w_{i_n}^n h(u_{i_1}^1,...,u_{i_n}^n)$$

An exception is a case where pdf f can be factored into the multiplication of n of one-dimensional pdf after the variable transformation has been carried out. A variate multiplication rule can be formed by combining a set of one-dimensional Gauss rules. A multiplication rule has $N=\prod_{j=1}^{M} J_j$ point, where J_j is the number of points used along the axes jth. Lobatto rules are spherical for the reintegration of normal multivariate distributions that only require $N=2^{M+1}-1$ point so that the actual integration of all polynomials of five or more degrees (Jondeau et al., 2007).

3. Settlement Approximation Based on Moment

Solving general problems, which may not be simple, we may focus on the approximation of this problem based on high moments.

3.1. Approximate Utility Utilities

Since we have been primarily interested in measuring the effect of high moments on asset allocation, we are now approximating the utility of hope by expanding the Taylor series around wealth of hope. In this context, the utility function can be approximated by the following form, when the expansion is formed above the fourth degree (to alleviate, we ignore the time index)

$$U(W)=U(W)+U^{(1)}(\overline{W})(W-\overline{W})+\frac{1}{2}U^{(2)}(\overline{W})(W-\overline{W})^2$$
$$+\frac{1}{3!}U^{(3)}(\overline{W})(W-\overline{W})^3+\frac{1}{4!}U^{(4)}(\overline{W})(W-\overline{W})^4+\varepsilon$$

where $\overline{W}=E[W]$ and ε is the rest of Taylor's. The expectation utility is simply approximated by

$$E[U(W)] \approx H(\overline{W})+\frac{1}{2}U^{(2)}(\overline{W})\sigma^2(W)+\frac{1}{3!}U^{(3)}(\overline{W})s^3(W)+\frac{1}{4!}U^{(4)}(\overline{W})k^4(W) \qquad (5)$$

Where $\sigma^2(W)$, $s^3(W)$, and $k^4(W)$ respectively are quantities $E[W-\overline{W}]^j$ for $j=1,...,4$.

In the case of HOW utility functions (2), the approximation to utility expectations is given by

$$E[U(W)] \approx -\exp(-\lambda\overline{W})\left[1+\frac{\lambda^2}{2}\sigma^2[W]-\frac{\lambda^3}{6}s^3[W]+\frac{\lambda^4}{24}k^4[W]\right]$$

or, in terms of portfolio income moments

$$E[U(W)] \approx -\exp(-\lambda\mu_p)\left[1+\frac{\lambda^2}{2}\sigma_p^2-\frac{\lambda^3}{6}s_p^3+\frac{\lambda^4}{24}k_p^4\right] \tag{6}$$

Where μ_p, σ_p^2, s_p^3, and k_p^4 respectively declare expected income, variance, skewness, and kurtosis of portfolio income.

After some clear simplifications, FOC can be defined successively as

$$(\mu-r_f)=\frac{\frac{\lambda}{2}\frac{\partial\sigma_p^2}{\partial\alpha}-\frac{\lambda^2}{6}\frac{\partial s_p^3}{\partial\alpha}+\frac{\lambda^3}{24}\frac{\partial k_p^4}{\partial\alpha}}{1+\frac{\lambda^2}{2}\sigma_p^2-\frac{\lambda^3}{6}s_p^3+\frac{\lambda^4}{24}k_p^4} \tag{7}$$

Optimal portfolio weights can be alternatively obtained by maximizing form (26) or by solving equation (7). Investigation of relations (7) shows that calculating this form would be simpler if the variance, skewness, and kurtosis of portfolio income and derivatives are known (Denuit et al., 2005).

We can also pay attention to the rank of utility functions (or CRRA, for Constant Relative Riskversion), defined as

$$E[U(W)]=\begin{cases}\frac{W^{1-\gamma}}{1-\gamma};\ \text{jika } \gamma>1\\ \log(W);\ \text{jika } \gamma=1\end{cases} \tag{8}$$

where γ the size of the investor's relative risk aversion constant (CRRA). In contrast to the HOW utility, CRRA does not converge to an asymptote for an increase in wealth.

In the case of the CRRA utility function, using the form (5), we obtain the following approximation for the utility of expectation

$$\begin{aligned}E[U(W)] \approx \frac{\overline{W}^{1-\gamma}}{1-\gamma}-\frac{\gamma}{2}\overline{W}^{-\gamma-1}\sigma^2(W)+\frac{\gamma(\gamma+1)}{6}\overline{W}^{-\gamma-2}s^3(W)\\ -\frac{\gamma(\gamma+1)(\gamma+2)}{24}\overline{W}^{-\gamma-3}k^4(W)\end{aligned} \tag{9}$$

or, in portfolio income terms

$$E[U(W)] \approx \frac{\mu_p^{1-\gamma}}{1-\gamma}\left[1-\frac{\gamma}{2\mu_p^2}\sigma_p^2+\frac{\gamma(\gamma+1)}{6\mu_p^3}s_p^3-\frac{\gamma(\gamma+1)(\gamma+2)}{24\mu_p^4}k_p^4\right] \tag{10}$$

The FOC for the CRRA function can be rewritten, after simplification

$$(\mu-r_f)=\frac{\frac{\gamma}{2(1+\mu_p)}\frac{\partial\sigma_p^2}{\partial\alpha}-\frac{\gamma(\gamma+1)}{6(1+\mu_p)^2}\frac{\partial s_p^3}{\partial\alpha}+\frac{\gamma(\gamma+1)(\gamma+2)}{24(1+\mu_p)^3}\frac{\partial k_p^4}{\partial\alpha}}{1+\frac{\gamma(\gamma+1)}{2(1+\mu_p)^2}\sigma_p^2-\frac{\gamma(\gamma+1)(\gamma+2)}{6(1+\mu_p)^3}s_p^3+\frac{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{24(1+\mu_p)^4}k_p^4} \tag{11}$$

Optimal portfolio weights can be obtained alternatively by maximizing form (2.9) by solving equation (11).

3.2. Investor's Preferences Based on Moments

In the case of the CARA and CRRA utility functions, skewness weights and high moments in the expected utility are estimated depending on the risk-avoidance parameters. this is a preference for skewness and risk-avoidance for kurtosis. Therefore, moment jth weights are strongly related to derivatives jth of utility functions (Dowd, 2002; Jorion, 2002).

Usually, risk aversion investors are assumed to have utility functions with the first and second derivatives as follows

$$U^{(1)}(W) > 0,\ \forall W \tag{12}$$

$$U^{(2)}(W) < 0,\ \forall W \tag{13}$$

The first condition means that the tool has a positive marginal utility for wealth, for example, unpleasant concerning wealth. The second condition is that the decline in marginal utility with wealth affects risk avoidance.

To investigate preferences concerning high moments, the following definitions are very useful.

Definition 1. An investor is consistent in the direction of preference for the moment that has a utility function for which the derivative has the same sign whatsoever. The investor is loudly consistent in the direction of preference for the moment if the derivative of- has an exact inequality concerning zero.

Then we have the following theorem

Theorem 2 (Scott and Horvath, 1980). Investors with the positive marginal utility of wealth for all levels of wealth (condition (2.12)), risk aversion consistent at all levels of wealth (condition (2.13)), and hard consistency of preference moments, will have positive preferences for positive skewness, ie

$$U^{(3)}(W) > 0,\ \forall W\ . \tag{14}$$

Theorem 3 (Scott and Horvath, 1980). Consistent risk retention (condition (13)), hard consistency of moment preference and positive preference for positive skewness (condition (14)) imply a negative preference for kurtosis, i.e.

$$U^{(4)}(W) < 0,\ \forall W\ .$$

In general, positive marginal utility assumptions, consistent risk avoidance together with hard consistency imply moment preference

$$U^{(n)}(W) > 0,\ \forall W \text{ if } n \text{ is odd, and}$$

$$U^{(n)}(W) < 0,\ \forall W \text{ if } n \text{ is even.}$$

Finally, the assumption means that $-U^{(3)}(W)/U^{(2)}(W)$ is the decrease in W, so $U^{(4)}(W) < 0,\ \forall W$.

3.3. Portfolio Moment Calculation

For a variable *n* system, the dimensions of the covariance matrix are (n,n), but only $n(n+1)/2$ the elements are calculated. Using the same method, the co-skewness matrix has dimensions (n,n,n), but only $n(n+1)(n+2)/6$ elements are calculated. Finally, the co-kurtosis matrix has dimensions (n,n,n,n), but only $n(n+1)(n+2)(n+3)/24$ elements are counted. For $n=5$, there will be 15 different elements for the covariance matrix, 35 elements for the co-skewness matrix, and 70 elements for the co-kurtosis matrix. Define co-skewness and co-kurtosis between asset income as

$$s_{ijk} = E[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)]^{1/3},$$

and

$$k_{ijkl} = E[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)(r_l - \mu_l)]^{1/4}.$$

As recommended by Athayde and Flores (2004), we transform the co-skewness matrix into a matrix, simple by partitioning each layer and once, in the same order, sideways. For example, in the case of assets, it produces a co-skewness matrix

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right]$$

$$= \left[s_{1jk} \quad s_{2jk} \quad s_{3jk} \right]_{j,k=1,2,3}$$

Where s_{1jk} is the short notation for the matrix (n,n), $(s_{1jk})_{j,k=1,2,3}$. This notation is the development of the covariance matrix, denoted by M_2. In the same way, the co-kurtosis matrix is

$$M_4 = \left[k_{11kl} \quad k_{12kl} \quad k_{13kl} \middle| \quad \ldots \quad \middle| k_{31kl} \quad k_{32kl} \quad k_{33kl} \right]_{k,l=1,2,3}.$$

We can now define portfolio moments in various ways. To provide a portfolio weight vector α, income unconditional expectations, variance, skewness, and kurtosis of a portfolio, respectively

$$\mu_p = \alpha' \mu,$$

$$\sigma_p^2 = \alpha' M_2 \alpha,$$

$$s_p^3 = \alpha' M_3 (\alpha \otimes \alpha),$$

$$k_p^4 = \alpha' M_4 (\alpha \otimes \alpha \otimes \alpha),$$

which symbolizes Kronecker's multiplication. The derivative moment of a portfolio against is very easily calculated (Denuit et al., 2005; Clientbaum et al., 1988)

$$\frac{\partial \sigma_p^2}{\partial \alpha} = 2M_2 \alpha,$$

$$\frac{\partial s_p^3}{\partial \alpha} = 3M_3 (\alpha \otimes \alpha),$$

$$\frac{\partial k_p^4}{\partial \alpha} = 4M_4 (\alpha \otimes \alpha \otimes \alpha).$$

In general, if $m_p = (\sigma_p^2, s_p^2, k_p^2, \ldots)$ we state the central portfolio moment vector, we will find $m_{p,i}^i = \alpha' M_i (\alpha \otimes^i \alpha)$ where $\alpha \otimes^i \alpha = \alpha \otimes (\alpha \otimes^{i-1} \alpha)$ and $\alpha \otimes^1 \alpha = \alpha \otimes \alpha$. Next, it will be obtained $\partial m_{p,i}^{i} / \partial\alpha = iM_i(\alpha \otimes^i \alpha)$.

3.4. Optimal Portfolio Allocation

Now how to complete the allocation of assets, when high moments are included in the problem of optimization. Athayde and Flores (2004) make a quasi-analytic solution to the problem of efficient portfolios by defining moments as tensors and then solving optimization problems. The solution, using non-linear optimization methods.

Equation (2.11) can be rewritten as

$$(\mu - r_f) - \delta_1(\alpha)[M_2\alpha] + \delta_2(\alpha)[M_3(\alpha \otimes \alpha)] - \delta_3(\alpha)[M_4(\alpha \otimes \alpha \otimes \alpha)] = 0 \tag{15}$$

Where δ_1, δ_2, and δ_3 are nonlinear functions of α. As many n of these equations can be easily solved numerically, using a standard optimization package (Denuit et al., 2005; Mood et al., 1963).

4. Conclusion

In optimizing portfolio asset allocation where returns are non-normal distributed, it can be done by direct maximization of utility expectations, and approximation of settlement based on moments. The direct maximization approach to expectation utility assumes that investors have a utility function that depends on the wealth of expectations for the coming period, and in certain cases, exponential or quadratic utility functions are often used. Approximate settlement based on the moment, measuring the effect of the high moment on asset allocation is used to estimate the utility of expectations by expanding the Taylor series around the wealth of expectations. The expected utility of this expectation will involve averages, variances, skewness, and kurtosis, and can be extended to even higher moments.

References

Batuparan, S. D. (2001). Kerangka Kerja Risk Management. (www.bexi.co.id/images/_res/perbankan-Kerangka Kerja Risk Management.pdf, accessed on 2 Maret 2008).

Beronilla, L., Nikkin, & Mapa, S. D. (2007). Range-Based GARCH: A New Method of Estimating Value-at-Risk. (www.nscb.gov.ph/ncs/10thNCS/abstracts/Contributed/36_Forecasting/abstract_mapa.pdf, accessed on 2 Maret 2008).

Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R. & Laeven, R. (2005). Risk Measurement With Equivalent Utility Principles. (*Download from internet January 2007*).

Dowd, K. (2002). *An Introduction to Market Risk Measurement.* New Delhi, India: John Wiley & Sons, LTD.

Jondeau, E., Poon, SH. & Rockinger, M. (2007). *Financial Modeling Under Non-Gaussian Distribution.* London: Springer-Verlag.

Jorion, P. (2002). *Value at Risk*: The New Benchmark for Managing Financial Risk. Second Edition. Singapore: McGraw-Hill Co.

Mood, A. M., Graybill, F. A. & Boes, D. C. (1963). Introduction to the Theory of Statistics, Third Edition. New

York: McGraw Hill Book Company.

Ruppert, D. (2004). *Statistics and Finance* : An Introduction. New York : Springer-Verlag.

Sukono, Subartini, B., Napitupulu, H., Hidayat, Y., Putra, A.S., & Bon, A.T. (2019). Value-at-Risk and Modified Value-at-Risk under Asset Liability by Using Time Series Approach. *Proceedings of the International Conference on Industrial Engineering and Operations Management*, Bangkok, Thailand, March 5-7, 2019, 2106-2117.