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On Quasi-Newton Method for Solving Fuzzy Nonlinear Equations

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Abstract

This paper presents Quasi Newton's (QN) approach for solving fuzzy nonlinear equations. The method considers an approximation of the Jacobian matrix which is updated as the iteration progresses. Numerical illustrations are carried, and the results shows that the proposed method is very encouraging.

Keywords: Quasi-Newton's method, Broyden's method, fuzzy nonlinear equations, parametric form

1. Introduction

Systems of nonlinear equations of the form

$$F(x) = 0 \tag{1}$$

where $F: R^n \rightarrow R^n$ is a real-valued function of a vector, is widely used in areas such as engineering, mathematics, computer science and social science. The concept of fuzzy numbers and arithmetic operations involving these numbers were due to Zadeh (1965) and the famous application of fuzzy number arithmetic is nonlinear equations whose parametric form are all or partially represented by fuzzy numbers (Abbasbandy and Asady, 2004; Kajani et al, 2005; Waziri and Moyi, 2016). Also, existing literature that uses standard analytic technique like Buckley and Qu (1990) are not suitable for solving systems such as

(1) $ax^5 + by^4 + cx^3 + dy^3 + ex^2y^2 + f = g$
(2) $x - cosy = p$

where a, b, c, d, e, f, g and p are fuzzy numbers. Here we consider these equations, in general, as

$$F(x) = c$$

To handle some of the pitfalls identified in (Buckley and Qu, 1990), many numerical methods have been introduced (Abbasbandy and Asady 2004; Waziri and Majid, 2012; Sulaiman et al, 2018) for example, Abbasbandy and Asady (2004) used Newton's method for solving fuzzy nonlinear equations. The method requires computing and storing the Jacobian matrix in every iteration. This leads to modification of Newton's method in order to reduce computation burden. Amirah et al (2010), apply Broyden's method for solving fuzzy nonlinear equations. Mustafa Mamat et al (2014), employ trapezoidal Broyden method for solving systems of nonlinear equations. Waziri and Moyi (2016), used chord method to solve dual fuzzy nonlinear equations. Kelley (1995) used Shamanskii-like method to solve nonlinear equations at singular point and Sulaiman et al (2018); Sulaiman et al (2018) further apply the Shamanskii's approach for fuzzy nonlinear problems. These methods do not evaluate the Jacobian at each iteration. Sulaiman et al (2018), employ the Conjugate Gradient method to solve fuzzy nonlinear equations. In this paper, we consider a Broyden's-like method for solving systems of fuzzy nonlinear equations. This method can best be described as belonging to the family of Quasi-Newton's method. The anticipation has been to reduce the computational burden of the Jacobian matrix in every iteration.

The paper is structured as follows. In section 2, we present some basic definition and fundamental results of fuzzy numbers. In section 3, we present the Broyden's-like method for solving nonlinear equation. In section 4, we present a Broyden's-like method for solving fuzzy nonlinear equation. In section 5, we illustrate our method by some numerical examples. Conclusions are given in the last section.

2. Preliminaries

This section presents some basic definition of fuzzy numbers.

Definition 1: [18]
A fuzzy number is a set like $u: R \to I = [0,1]$ which satisfy the following conditions,

(1) u is upper semicontinuous,
(2) $u(x) = 0$ outside some interval $[c, d]$,
(3) There are real numbers a, b $such$ $that$ $c \leq a \leq b \leq d$ and
    (3.1) $u(x)$ is monotonic increasing on $[c, a]$
    (3.2) $u(x)$ is monotonic decreasing on $[b, d]$
    (3.3) $u(x) = 1, a \leq x \leq b.$

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is also given in (Goetschel and Voxman, 1986) as follows.

Definition 2. (Dubois and Prade, 1980)
A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of function $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfies the following requirements:

(1) $\underline{u}(r)$ is bounded monotonic increasing left continuous function,
(2) $\overline{u}(r)$ is bounded monotonic decreasing left continuous function,
(3) $\underline{u}(r) < \overline{u}(r), 0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, 0 \leq r \leq 1$ (Fang, 2002; Peeva, 1992). A popular fuzzy number is the trapezoidal fuzzy number $u = (x_0, y_0, \alpha, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness α and right fuzziness β where the membership function is

$$u(x) = \begin{cases} \frac{1}{\alpha}(x - x_0 + \alpha), & x_0 - \alpha \leq x \leq x_0, \\ 1 & x \in [x_0, y_0] \\ \frac{1}{\beta}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

and its parametric form is

$$\underline{u}(r) = x_0 - \alpha + \alpha r, \ \overline{u}(r) = y_0 + \beta - \beta r.$$

Let $TF(R)$ be the set of all triangular fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v}) and \ k > 0$ we define $u + v$ and multiplication by real number $k > 0$ as

$$(\underline{u + v})(r) = \underline{u}(r) + \underline{v}(r), \ (\overline{u + v})(r) = \overline{u}(r) + \overline{v}(r),$$
$$(\underline{ku})(r) = k\underline{u}(r), \qquad (\overline{ku})(r) = k\overline{u}(r).$$

3. Broyden's Method

Broyden's method belongs to a class of methods known as Quasi-Newton methods that are designed to improve Newton's method in terms of efficiency (Broyden, 1965). The method tries to approximate the Jacobian $J'(x^{(k)})$ or its inverse $J'(x^{(k)})^{-1}$ by $H_k = B_k^{-1}$, which is updated as the nonlinear iteration progresses. The update is done via secant approximation to the derivative Dennis, 1983). Given an initial guess x_0 the Broyden method generate a sequence of point $\{x_k\}$ in the form

$$x_{k+1} = x_k - B_k^{-1} F(x_k) \qquad k = 0,1,2 \ldots \tag{2}$$

where B_k is an approximation to the Jacobian (matrix of the partial derivative of $f^{(k)}$ evaluated at $x^{(k)}$). Choose B_{k+1} in such a way that

$$B_{k+1}d_k = y_k \tag{3}$$

where $y_k = F\left(x^{(k+1)}\right) - F(x^{(k)})$ and $d_k = x^{(k+1)} - x^{(k)}$. The above equation is referred to as Quasi-Newton condition(Broyden, 1965; Kelley,1995). However, for single rank methods, B_{k+1} is chosen satisfying (3) such that

$$B_{k+1} = B_k - \frac{B_k(y_k)+B_kF(x_k))d_k^T}{d_k^T(y_k)} \tag{4}$$

where $d_k^T(y_k) \neq 0$. If $A(x_k) = A_k = B_k^{-1}$), then

$$A_{k+1} = A_k - (y_k - F(x_k))d_k^T A_k / d_k^T F(x_k) \tag{5}$$

In this study, we use the approximate $J\left(\underline{x}_0, \overline{x}_0, r\right) = B_0(r)$ and B_0^{-1} to solve (1) with initial guess chosen close to the exact solution and we the update for B_{k+1} as

$$B_{k+1} = B_k + \frac{(d_1 - B_k D_1)d_k}{d_k, D_1} \tag{6}$$

where $d_1 = (x - x_0)$, $D_1 = (F - F_0)$, and $d_k = (d_1^T, B_k)$.

We now state the convergence theorems of the Broyden's method.

Theorem 1 (Kelley,1995)
Let the standard assumptions hold. Then there are δ and δ_B such that if $x_0 \in B(\delta)$ and $\|E_0\|_2 < \delta_B$, the Broyden sequence for the data (F, x_0, B_0) exist and $x_k \rightarrow x^*$ q-superlinearly.

Theorem 2 (Kelley,1995)
Let the standard assumptions hold and $r \in (0,1)$ be given. Then there are δ and δ_B such that if $x_0 \in B(\delta)$ and $\|E_0\|_2 < \delta_B$ the Broyden sequence for the data (F, x_0, B_0) exist and $x_k \rightarrow x^*$ q-linearly with q-factor at most r.

Refer to (Kelley,1995)) for the theoretical proof of the above theorems.

4. Iterative approach for solving fuzzy nonlinear equations

In this section, we intend to obtain a solution for fuzzy nonlinear equation

$$F(x) = 0.$$

whose parametric form is as follows:

$$\underline{F}\left(\underline{x}, \overline{x}; r\right) = 0$$

$$\overline{F}\left(\underline{x},\overline{x};r\right) = 0.\ \forall r \in [0,1]\,. \tag{7}$$

Assume that $\alpha = (\underline{\alpha},\overline{\alpha})$ is the solution to the nonlinear system (7), i.e.

$$\underline{F}\left(\underline{\alpha},\overline{\alpha};r\right) = 0,$$
$$\overline{F}\left(\underline{\alpha},\overline{\alpha};r\right) = 0,\ \forall r \in [0,1]$$

Now, if $x_0 = (\underline{x}_0,\overline{x}_0)$ is an approximate solution for this nonlinear system, then $\forall r \in [0,1]$, there are $h(r), k(r)$ such that

$$\underline{\alpha}(r) = \underline{x}_0(r) + h(r),$$
$$\overline{\alpha}(r) = \overline{x}_0(r) + k(r).$$

By applying the Taylor series of $\underline{F},\overline{F}$ about $(\underline{x}_0,\overline{x}_0)$, then $\forall r \in [0,1]$,

$$\underline{F}\left(\underline{\alpha},\overline{\alpha};r\right) = \underline{F}\left(\underline{x}_0,\overline{x}_0,r\right) + h\,\underline{F}_{\underline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) + g\,\underline{F}_{\overline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) + 0(h^2+hk+h^2) = 0$$

$$\overline{F}\left(\underline{\alpha},\overline{\alpha};r\right) = \overline{F}\left(\underline{x}_0,\overline{x}_0,r\right) + h\,\overline{F}_{\underline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) + g\overline{F}_{\overline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) + 0(h^2+hk+h^2) = 0.$$

However, suppose $\underline{x}_0$ and $\overline{x}_0$ are near to $\underline{\alpha}\ and\ \overline{\alpha}$, respectively, then $h(r)\ and\ k(r)$ are small enough. Let us assume that all needed partial derivatives exist are bounded. Therefore for enough small $h(r)\ and\ k(r)$, where $\forall r \in [0,1]$, we have,

$$\underline{F}\left(\underline{x}_0,\overline{x}_0,r\right) + h\,\underline{F}_{\underline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) + g\,\underline{F}_{\overline{x}}\left(\underline{x}_0,\overline{x}_0;r\right) = 0$$

$$\overline{F}\left(\underline{x}_0,\overline{x}_0,r\right) + h\,\overline{F}_{\underline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) + g\overline{F}_{\overline{x}}\left(\underline{x}_0,\overline{x}_0;r\right) = 0.$$

Hence, $h(r)$ and $k(r)$ are unknown quantities that can be obtained by solving the following equations, $\forall r \in [0,1]$,

$$J(\underline{x}_0,\overline{x}_0,r)\ \begin{pmatrix} h(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} -\underline{F}(\underline{x}_0,\overline{x}_0,r) \\ -\overline{F}(\underline{x}_0,\overline{x}_0,r) \end{pmatrix} \tag{8}$$

where

$$J(\underline{x}_0,\overline{x}_0,r) = \begin{bmatrix} \underline{F}_{\underline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) & \underline{F}_{\overline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) \\ \overline{F}_{\underline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) & \overline{F}_{\overline{x}}\left(\underline{x}_0,\overline{x}_0,r\right) \end{bmatrix}$$

is the Jacobian matrix of the function $F = (\underline{F},\overline{F})$ evaluated in $x_0 = \left(\underline{x}_0,\overline{x}_0\right)$. However, $J(\underline{x}_0,\overline{x}_0,r)$ in (8) is derived by a derivative estimation $J(\underline{x}_k,\overline{x}_k,F(x_k),r)$ for $k = 0,1,2$ ... and for all $r \in [0,1]$

Hence, the next approximations for $\underline{x}(r) and\ \overline{x}(r)$ are as follows

$$\underline{x}_1(r) = \underline{x}_0(r) + h(r),$$

$$\overline{x}_1(r) = \overline{x}_0(r) + k(r),$$

for all $r \in [0,1]$.

We can obtain approximated solution, $r \in [0,1]$, by using the recursive scheme

$$\begin{aligned}\underline{x}_{n+1}(r) &= \underline{x}_n(r) + h_n(r),\\ \overline{x}_{n+1}(r) &= \overline{x}_n(r) + k_n(r),\end{aligned} \tag{9}$$

when $n = 1,2, \ldots$ Analogous to (5)

$$J(\underline{x}_n, \overline{x}_n, r)\ \begin{pmatrix} h(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} -\underline{F}(\underline{x}_0, \overline{x}_0, r) \\ -\overline{F}(\underline{x}_0, \overline{x}_0, r) \end{pmatrix}$$

Now, if $J(\underline{x}_n, \overline{x}_n, r)$ is nonsingular, then from (8) we obtain the recursive scheme of Newton's method as follows,

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - J(\underline{x}_n, \overline{x}_n, r)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n, r) \\ \overline{F}(\underline{x}_n, \overline{x}_n, r) \end{bmatrix}$$

Now, we present the algorithm for our proposed approach (Newton-Broyden's Method) as follows:

Algorithm 1: Newton-Broyden's Method (NBM)

Step 1. Transform the fuzzy nonlinear equations into parametric form
Step 2. Determine the initial guess x_0 by solving the parametric equations for $r = 0$ and $r = 1$. For $k = 0,1,2 \ldots$
Step 3. Compute the initial Jacobian matrix

$$J(\underline{x}_0, \overline{x}_0, r) = B_0(r)$$

Step 4. Compute $B_0(r)s_k = -F(x_k)$
Step 5. Compute $B_1(r) = B_0(r)^{-1}$
Step 6. Compute the update (NBM) by (6)
Step 7. Repeat step 3 to step 6 and continue with the next *k* until $\epsilon \leq 10^{-4}$ are satisfied.
.

5. Numerical Results

In this section, two examples where considered to illustrate the performance of the proposed method for solving fuzzy nonlinear equation. Also, the solutions were plotted in Figure 1 and Figure 2 respectively. The considered benchmark problems are taken from (Waziri and Majid, 2012).

Example 1. Consider a fuzzy nonlinear equation

$$(6,2,2)x^2 + (2,1,1)x = (2,1,1)x^2 + (2,1,1)$$

Without any loss of generality, let x be positive, then the parametric form of this equation is as follows:

$$(4+2r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)\underline{x}^2(r) + (1+r)$$
$$(8-2r)\overline{x}^2(r) + (3-r)\overline{x}\,(r) = (3-r)\overline{x}^2(r) + (3-r)$$

Rewriting yields

$$(3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)$$
$$(5-r)\overline{x}^2(r) + (3-r)\overline{x}\,(r) = (3-r)$$

Let $J(\underline{x},\overline{x},r) = B_0(r)\ and\ J(\underline{x},\overline{x},r)^{-1} = B_1(r)^{-1}$
Then,

$$B_0(r) = \begin{bmatrix} 2(3+r)\underline{x}(r) + (1+r)(r) & 0 \\ 0 & 2(5-r)\overline{x}(r) + (3-r) \end{bmatrix}$$

and

$$B_1(r)^{-1} = \begin{bmatrix} \frac{1}{2(3+r)\underline{x}(r) + (1+r)(r)} & 0 \\ 0 & \frac{1}{2(5-r)\overline{x}(r) + (3-r)} \end{bmatrix}$$

To obtain the initial guess, we let $r = 0$ and $r = 1$ in the above system, therefore
$r = 1$

$$4\underline{x}^2(1) + 2\underline{x}(1) = 2$$
$$4\overline{x}^2(1) + 2\overline{x}\,(1) = 2$$

$r = 0$

$$3\underline{x}^2(0) + \underline{x}(0) = 1$$
$$5\overline{x}^2(0) + 3\overline{x}\,(0) = 3$$

We have, $\underline{x}(0) = 0.4343, \overline{x}(0) = 0.5307\ and\ \underline{x}(1) = \overline{x}(1) = 0.5000$. Considering, initial guess as $x_0 = (0.4,0.5)$, after three iterations, we obtain the solution with the maximum error less than 10^{-5}. The performance profile is given in Figure 1.

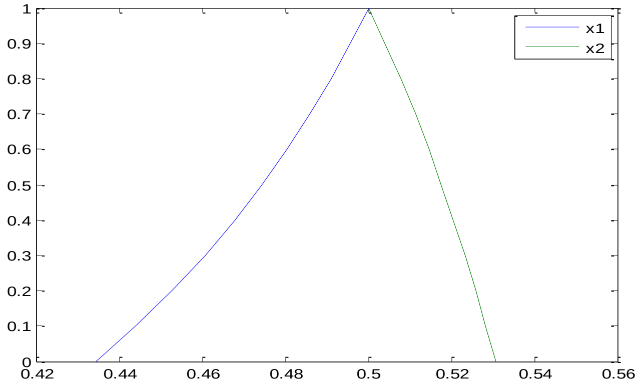

Figure 1: interactive solution of example 1

Example 2. Consider a fuzzy nonlinear equation

$$(2,1,1)x^3 + (3,1,1)x^2 + (4,1,1)x = (4,1,1)x + (4,2,4)$$

Without any loss of generality, we assume that x be positive, then we have the parametric equation as follows:

$$(1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) = (2+2r)$$
$$(3-r)\overline{x}^3(r) + (4-r)\overline{x}^2\,(r) = (8-4r)$$

Let $J(\underline{x},\overline{x},r) = B_0(r)\ and\ J(\underline{x},\overline{x},r)^{-1} = B_1(r)^{-1}$. Then,

$$B_0(r) = \begin{bmatrix} 3(3+r)\underline{x}^2(r) + 2(2+r)\underline{x}(r) & 0 \\ 0 & 3(3-r)\underline{x}^2(r) + 2(4-r)\underline{x}(r) \end{bmatrix}$$

and

$$B_1(r)^{-1} = \begin{bmatrix} \frac{1}{3(3+r)\underline{x}^2(r) + 2(2+r)\underline{x}(r)} & 0 \\ 0 & \frac{1}{3(3-r)\underline{x}^2(r) + 2(4-r)\underline{x}(r)} \end{bmatrix}$$

Let $r = 0$ and $r = 1$. We then obtain the initial guess as follows

$$\underline{x}^3(0) + 2\underline{x}^2(0) = 2$$
$$3\overline{x}^3(0) + 4\overline{x}^2\,(0) = 8$$

and

$$2\underline{x}^3(1) + 3\underline{x}(1) = 4$$
$$2\overline{x}^3(1) + 3\overline{x}\,(1) = 4$$

using $r = 0\ and\ r = 1$, to solve the above system, we obtain the initial guess $x_0 = (0.9, 0.9, 0.15)$ we obtain the solution after four iterations with maximum error less than 10^{-5}.
The performance profile is given in Figure 2.

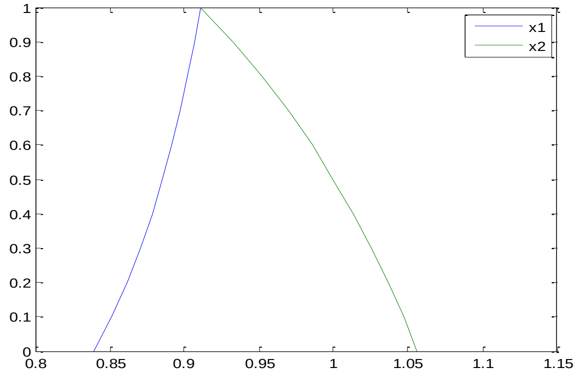

Figure 2: interactive solution of example 2

6. Conclusion

In this paper, we suggested a numerical method for solving fuzzy nonlinear equations instead of standard analytical technique. The fuzzy nonlinear equations are written in parametric form and then solved via Quasi Newton's method. Finally, numerical examples were presented to illustrate the proposed method and the numerical results shows that our method is very effective.

References

Abbasbandy, S and Asady, B. (2004) Newton's method for solving fuzzy nonlinear equations, *Appl. Math. Comput*, 156, 381-386.

Amirah, R. Lazim, M., and Mamat, M. (2010). Broyden's method for solving Fuzzy nonlinear equations. *Advances in fuzzy system*, Article ID 763270, 6 pages.

Buckley, J. J. and Qu, Y. (1990) Solving linear and quadratic fuzzy equation. *Fuzzy Set Syst*. 38, 43-59.

Broyden, C. G. (1965). A class of methods for solving nonlinear simultaneous equations. *Mathematics of computation*, 19(92), 577-593.

Dennis, J, E. (1983). Numerical methods for unconstrained optimization and nonlinear equations. Prince-Hall, Inc., Englewood Cliffs, New Jersey, USA.

Dubois, D and Prade, H. (1980). Fuzzy Sets and Systems: Theory and Application, Academic Press, New York, USA.

Fang, J. (2002). On nonlinear equations for fuzzy mappings in probabilistic normed spaces. *Fuzzy Sets and Systems*, 131, 357-364.

Goetschel Jr, R. and Voxman, W. (1986). Elementary fuzzy calculus. *Fuzzy set and Systems,* 18, 31-43.

Kajani, M. T., Asady, B., and Vencheh, A. H. (2005). An iterative method for solving dual fuzzy nonlinear equations. *Applied Mathematics and Computation*, 167, 316-323.

Kelley, C. T. (1995). Iterative Methods for Linear and Nonlinear Equations". SIAM, Philadelphia, PA.

Mamat, M., Muhammad, K., and Waziri, M. Y. (2014) Trapezoidal Broyden's method for solving systems of nonlinear equations. *Applied Mathematical Sciences,* 8, 251-60.

Peeva, K. (1992) Fuzzy linear systems. *Fuzzy Sets Syst*, 49, 339-355.

Sulaiman, I. M., Mamat, M., and Waziri, M. Y. (2018). Shamanskii method for Solving Fuzzy Nonlinear Equation. *Proc. of the 2nd IEEE Intl Conf on Intl Syst Engr (ICISE), Malaysia*. pp. 20-21.

Sulaiman, I. M., Mamat, M., Mohamed, M. A., and Waziri, M. Y. (2018). Diagonal Updating Shamasnkii-Like method for Solving Fuzzy Nonlinear Equation. *Far East Journal of Math Sci*, 103(10), 1619-1629.

Sulaiman, I. M., Mamat, M., Waziri, M. Y., Mohamed, M. A., and Mohamad, F. S. (2018). Solving Fuzzy Nonlinear Equation via Levenberg-Marquardt Method. *Far East Journal of Math Sci*, 103(10), 1547-1558.

Sulaiman, I. M., Waziri, M. Y., Olowo, E. S., and Talat, A. N. (2018) Solving Fuzzy Nonlinear Equations with a New Class of Conjugate Gradient Method. *Malaysian Journal of Computing and Applied Mathematics*, 1(1), 11-19.

Waziri. M. Y. and Majid, Z. A. (2012). A new approach for solving dual Fuzzy nonlinear equations. *Advances in Fuzzy Systems*. Article ID 682087, 5 pages doi:10.1155/2012/682087 no. 25, 1205 - 1217.

Waziri, M. Y. and Moyi, A. U. (2016). An alternative approach for solving dual fuzzy nonlinear equations. *International Journal of Fuzzy Systems*, 18(1), 103-107.

Zadeh, L. A. (1965) fuzzy sets. *Information and Control*, 8, 338-353.