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Graph Models of Harems and Tournaments in Sports Clubs

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Abstract

Looking at the extension of Hall's marriage theorem to harems, where some people are allowed to have more than one partner. Traditionally, in harems, any man can have multiple wives, but no woman can have more than one husband. then consider the different types of matches by looking at 'round robin tournaments' in sports clubs. An unexpected connection between the two worlds emerged when we were able to use our harem results to deduce theories about the tournament.

Keywords: Hall's marriage theorem, Hall's theorem - harem form, Landou theorem, tournamen.

1. Introduction

The key to using Hall's marriage theorem is to realize that, in essence, matching things comes up in lots of different ways. The key in thinking about Hall's marriage theorem is to realize that it means, in essence, the obvious matching condition is the only one we need. We'll start by discussing the first, then we'll discuss the second.

What is matching? The canonical example invovles marriage. We have several women who have a list of boys they would be happy to marry. Any boys would be happy to marry any girls who would be happy to marry him. Is it possible to make all the girls happy?

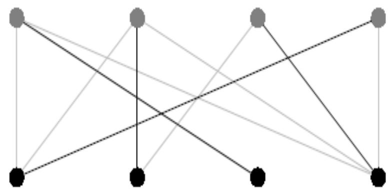


Figure 1: In real life, marriage is never as simple as this.

We can represent this as a graph, with girls as vertices on the top and boys as vertices on the bottom. Then we draw edges to show which girls like which boys. If we look at the darker edges, we see we can pick a bunch of marriages that makes all the girls happy.

Such a choice of marriage is called matching. The matching endpoints should be different: no boy is married to two or more girls, and no girl is married to two or more boys.

We say that a matching covers the set of girls if we can make all girls happy. A matching that would make all the boys happy covers the set of boys. A matching that makes them all happy covers both sets, and is called a perfect matching.

Basically, matching is pairing up two sets to each other. Other natural examples of matching are matching up robots to batteries that fit them, matching companies to people they want to hire, matching kids to gifts they want to receive. In these cases, we can figure out the two sets of vertices and when to draw an edge just fine (Wilson & Robin, 1996).

Now, matching things can come up in obvious ways, as above. Sometimes in a problem, we can see that it's asking for a matching, and we can just use Hall's to show a matching exists. But it can come up in non-obvious ways too, where finnding the matching is already half of the problem. Let's go back to our marriage matching. In our previous example, we managed to make everyone happy. But that's not always possible. Consider the graph below:

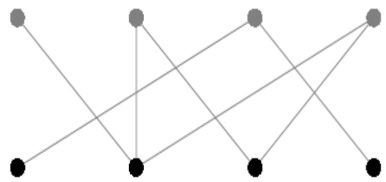


Figure 2: Ther’s somethink wrong here.

If you try to find a matching that makes everyone happy, you'll see that you can't. You just can't do it. What's stopping us from doing it? The more we try to find a matching, the more we see what's wrong with this diagram:

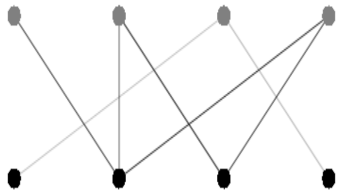


Figure 3: Even in simplified marriage thins can go wrong.

This is what's stopping us: these three girls. These three girls only like two boys. But we can't marry the two boys to three girls. So we can't make everyone happy, because at least one of these girls will be sad. Similarly, if we have two girls liking only one boy, we can't make everyone happy. If we have four girls liking only two boys, we can't make everyone happy. If we have n girls liking only m boys, where $n > m$; then we can't make everyone happy (Bryant & Victor, 1993).

This is called the matching condition. If we pick n girls, and they like m boys, then we should have $n \leq m$: Otherwise, if $n > m$; we can't fnd a matching. Here's an exercise (which you should do): look at the figure below and confirm that the matching condition holds for any subset of the women.

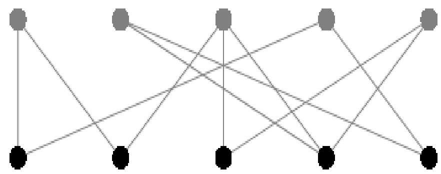


Figure 4: We can make everyone happy.

We can write this up in graph theoretical terms. We write $N(V)$ to denote the neighborhood of a set V: the set of all vertices adjacent to some vertex of V: In our example, our $N(V)$ for a subset V of girls would be the set of all men they like.

Then this is Hall's: (Hall's marriage theorem). Let G be a bipartite graph with bipartite sets X and Y. Then there exists a matching that covers X if and only if for each subset V of X,$|V| \leq |N(V)|$.

We call the condition, $|V| \leq |N(V)|$ for all subsets V of X, the matching condition. If the matching condition holds, a matching exists. We'll abuse terminology: we'll sometimes use matching condition to refer only to the inequality $|V| \leq |N(V)|$ itself (Wilson & Robin, 1996).

Landau proved that some rather obvious necessary conditions for a non-decreasing sequence of n integers to be the score sequence for some n tournament are, in fact, also sufficient (Landau, 1953).

Namely, the sequence is a score sequence if and only if, for each k, $1 \leq k \leq n$, the sum of the first k terms is at least $\binom{k}{2}$, with equality when $k = n$. There are now several proofs of this fundamental result in tournament theory, ranging from clever arguments involving gymnastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, to a constructive argument utilizing network flows and another one involving systems of distinct representatives (Reid, 1996).

2. Methods

2.1. Hall's theorem - marriage form

A set of girls can all choose a husband each from the boys that they know if and only if any subset of the girls (r of them, say) know between them at least r boys.

If the girls can all find husbands, then clearly any r girls must know between them at least r boys, namely their husbands.

Let the girls be named 1, 2, …, n and assume that any subset of them (r say) know between them at least r boys. We shall show by induction on m ($1 < m < n$) that the girls 1, 2, …, m can all find husbands from the boys that they know.

The case $m = 1$ is trivial since the girl 1 knows at least one boy whom she can then choose as her husband. So assume that $m > 1$, that the girls 1, 2, ..., m have found fiances (named 1', 2', ..., m' respectively) and that we now wish to find fiances for the girls 1, 2, ..., m and $m + 1$. These $m + 1$ girls know between them at least $m + 1$ boys and only m of those boys are engaged. So it is tempting to say that the girl named $m + 1$ must know some boy who is not engaged: if she does then we can find her a fiance. But (and this is what makes the result non-trivial) even though the girls 1, 2, ..., m, $m + 1$ know between them at least $m + 1$ boys, the girl $m + 1$ may personally only know boys from amongst 1', 2', ..., m', all of whom are already engaged. So how does she find herself a fiance?

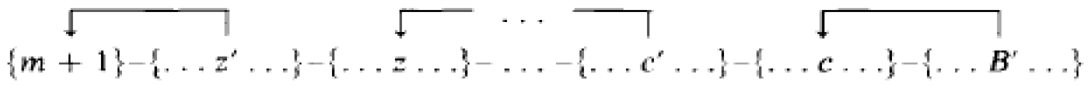
As in the previous example, let girl $m + 1$ throw a party. She invites all the boys she knows: they invite their fiancees, those girls invite all the boys they know who haven't already been invited, those boys invite their fiancees, those girls invite all the boys they know who haven't already been invited. This process continues until some boy (***B'*** say) is invited who is not already engaged.

Must that happen? Will some new unengaged boy eventually be invited? Let us represent the situation by some sets, where for ease of labelling we assume that the boys' `names' are chosen so that they are invited in numerical order:

$$\{m+1\} \text{ invites } \{1', \ldots, i'\} \text{ invites } \{1, \ldots, i\} \text{ invites } \{(i+1)', \ldots, j'\}$$
$$\text{invites} \ldots \text{ invites } \{(k+1)', \ldots, l'\} \text{ invites } \{k+1, \ldots, l\} \text{ invites } \{\ldots \boldsymbol{B}' \ldots\}.$$

How do we know that this process continues until some new unengaged boy is invited? If no new unengaged boys have been invited then at a typical stage in the process, when we have a set of newly invited girls $\{k + 1, \ldots, l\}$, whom will they invite next? So far the girls $m + 1$ and 1, 2, ..., l and the boys 1', 2', ..., l' have been invited (and that includes all the boys known to girls 1, 2, ... and k), but overall that's fewer boys than girls. So, by the given condition of girls knowing at least as many boys, there still remains to be invited a boy known to one of the girls there (and hence to one of $k + 1$, ..., l). Therefore the process will continue. There is only a finite number of engaged boys and so at some stage the process will include a boy B' who is not already engaged.

The party is now in full swing and the boy ***B'*** dances with one of the girls who invited him. Her fiance is a bit put out so he dances with a girl who invited him: her fiance is a bit annoyed so he dances with a girl who invited him: ... and (as in the previous example) this races its way back through the list of sets given above, eventually leading to a boy dancing with the girl $m + 1$:



Pairing off some engaged couples together with girl m + 1 and boy ***B'***

This dance is such a success that all the dancing couples break off all former commitments and get engaged. Any couples not dancing remain engaged to their previous partners. A few moments of thought shows you that the result of this process is to rearrange the engaged people dancing and girl $m + 1$ and boy ***B'*** into different engaged pairs. Hence overall we have found fiances for all the girls 1, 2, ..., $m + 1$ (Bryant & Victor, 1993).

Example In a group of seven boys and six girls of marriageable age

girl 1 knows boys 1', 2' and 3',
girl 2 knows boys 2' and 3',
girl 3 knows boys 3', 5' and 7',
girl 4 knows boys 1' and 2',
girl 5 knows boys 1', 2' and 3',
girl 6 knows boys 4', 5' and 6'.

Is it possible to find each of the girls a husband (i.e. a different boy for each from amongst those whom she knows)?

Solution It's easy to find husbands for the girls by trial and error, but we'll apply a process which will in fact work in general (as we shall see in the proof of the next theorem). We start to choose different boys for each of the girls in any way we like until we reach a girl for whom there is no boy left to choose. For example, I could get engaged to 1', 2 to 2', 3 to 3', 4 to 4' and 5 to 5'. But then girl 6 only knows 1' and 2', and they are already engaged. How can she find a partner?

Girl 6 throws a party. She invites all the boys she knows: they invite their fiancees: those girls invite all the boys they know who haven't already been invited: those boys invite their fiancees: those girls invite all the boys they know who haven't already been invited. This process continues until some boy (***B'*** say) is invited who is not already engaged. In this case it leads to

{6} invites {1', 2'} invites {1, 2} invites {3'} invites
{3} invites {4', 5'} invites {4, 5} invites {16', 7'}

We stop there because, for example, B' = 7' is not engaged. Now B' (7') dances with a girl he knows who invited him (4). Her fiance (4') is a bit annoyed so he dances with a girl who invited him (3): her fiance (3') is a bit put out so he dances with a girl who invited him (1): her fiance (1') is a bit miffed so he dances with a girl who invited him (6). In effect we are working back through the sets listed above:

Some engaged couples (1, 1'; 3, 3': 4, 4') together with girl 6 and boy 7' paired off into new couples

This smoochy dance is so successful that the dancing couples break off any former engagements and get engaged. No other couples are affected. That leaves

1 engaged to 3': 2 to 2': 3 to 4': 4 to 7': 5 to 5' and 6 to 1'

and we have managed to find husbands for all the girls.

2.2 Landau Theorem

Let $b_1, b_2. \dots, b_n$ be integers. Then they are scores in a tournament of n players if and only if

(i) $$b_1 + b_2 + \dots + b_n = \binom{n}{2}.$$

and

(ii) for $1 \le r \le n$ any r of the $b_i's$ add up to at least $\binom{r}{2}$.

Assume that there exists a tournament of *n* players with scores $b_1, b_2. \dots, b_n$. Then clearly the sum of these scores is the total number of games played, which is $\binom{n}{2}$. Also any r of the players will have played $\binom{r}{2}$ games amongst themselves and so they must have at least $\binom{r}{2}$ wins between them. Hence any *r* of the scores must add up to at least $\binom{r}{2}$. Thus properties (i) and (ii) are established.

Now let $b_1, b_2. \dots, b_n$ be integers satisfying (i) and (ii): then, in particular, taking *r* = 1 shows that these integers are non-negative. Now given any *s* of the numbers, $b_{i_1}, b_{i_2}, \dots, b_{i_s}$, say, consider the other $n - s$ numbers: they will add up to at least $\binom{n-s}{2}$ and so it follows that

$$b_{i_1} + b_{i_2} + \dots + b_{i_s} \le \binom{n}{2} - \binom{n-s}{2}.$$

Now imagine that young ladies 1, 2, ..., *n* enter a squash tournament. Offer as prizes $\binom{n}{2}$ handsome young men called '1, 2', '1, 3', ..., '1, *n*', '2, 3', '2, 4', ... and '$n - 1$, *n*': the boy 'i, j' is to be awarded as the prize in the game between players *i* and *j*. The girls get to know just the boys who they are playing for, so girl 2 for example gets to know the boys '1, 2', '2, 3', '2, 4', ... and '2, *n*'. Give any *s* of the girls, $i_1, i_2, \dots, i_s$, how many boys do they now know between them? They know all the boys except those exclusively known by the other $n - s$ girls; i.e. they know exactly $\binom{n}{2} - \binom{n-s}{2}$ boys between them. So, by the inequality displayed above, the *s* girls $i_1, i_2, \dots, i_s$ know between them at least $b_{i_1} + b_{i_2} + \dots + b_{i_s}$ boys. Hence by the harem form of Hall's theorem, proved earlier, the girls 1, 2, ..., *n* can find $b_1, b_2. \dots, b_n$ husbands, respectively, from amongst the boys they know. In other words the decisions about whether girl *i* or girl *j* should marry boy 'i, j' can be made so that girl 1 ends up with b_1, prizes, girl 2 with b_2 prizes, …. and girl *n* with b_n prizes. It follows that the results of all the games in the tournament can be fixed so that the number of prizes won by 1, 2, ..., *n* are $b_1, b_2. \dots, b_n$ respectively: these numbers are then the scores of the tournament, and the theorem is proved (Thommasen, 1981).

3. Result and Discustion

Hall's theorem - harem form. Let $b_1, b_2. \ldots, b_n$ be non-negative integers and let $G_1, G_2. \ldots, G_n$ be girls. Girl G_1 wants b_1 husbands (as always, from amongst the boys whom she knows), girl G_2 wants b_2 husbands, … and girl G_n wants b_n husbands. No boy can marry more than one girl. Then the girls' demands can all be satisfied if and only if any collection of the girls $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ knows between them at least $b_{i_1} + b_{i_2} + \ldots + b_{i_s}$ boys.

If the girls can all find the required numbers of husbands, then the girls $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ (as always, assumed to be distinct) must know between them at least $b_{i_1}, b_{i_2}, \ldots, b_{i_s}$ boys, namely their husbands.

Assume that any collection $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ of girls know between them at least $b_{i_1}, b_{i_2}, \ldots, b_{i_s}$ boys. Replace G_1 by b_1 copies of G_1 each of whom knows the same boys that G_1 did (so that, for example, if b_1 = 3 then G_1 is replaced by triplets). Similarly, replace G_2 by b_2 copies of G_2, ... and G_n by b_n copies of G_n, giving:

$$\text{girls: } \underbrace{G_1 \ldots G_1}_{b_1} \; \underbrace{G_2 \ldots G_2}_{b_2} \ldots \underbrace{G_n \ldots G_n}_{b_n}$$

We shall apply Hall's marriage theorem to find each girl one husband in this new situation. So we shall show that, in this new situation, any *r* girls know between them at least *r* boys. Choose any set of these new girls: let there be *r* of them, say, and assume these these *r* consist of at least one 'copy' of each of the girls $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$. Then clearly

$$r \leq b_{i_1} + b_{i_2} + \ldots + b_{i_s}$$

since the right-hand total is the number we would get if we took all copies of those girls. Now the given conditions tell us that in the original situation the girls $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$knew between them at least $b_{i_1}, b_{i_2}, \ldots, b_{i_s}$ boys (and hence at least *r* boys). It follows that our set of *r* girls (which includes a copy of each of $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$) in the new twin/triplet situation know between them at least *r* boys.

Hence, as any *r* girls do know at least *r* boys, we can apply Hall's marriage theorem to the new situation to find each of the girls a husband. So, for example, each of the b_1 copies of G_1 will get a husband: clearly these can be combined to give b_1 different husbands for the original girl G_1. Proceeding in this way gives each girl the required number of husbands in the original situation.

We now turn to the world of tournaments. In a squash club, for example, a'round robin tournament' for a group of players consists of each of them playing each of the others once, each game resulting in a win for one of the two players. Imitating this idea, a tournament of *n* players (called 1, 2, ... and *n*, say) consists of $\binom{n}{2}$ ordered pairs of players so that for $1 \leq i < j \leq n$ either the pair *ij* or the pair *ji* is included. The $\binom{n}{2}$ ordered pairs can be thought of as the results of the games between each pair of players, with the first number in the pair being the winner of that game: in this way we shall use all the normal sports jargon. In addition, a tournament of *n* players can be represented by a picture of the complete graph K_n with the vertices labelled 1, 2, ..., *n* and with an arrow on each edge, an arrow from *i* to *j* meaning that 'i beat *j*'.

Example The diagram on the below represents a tournament of the five players 1, 2, 3, 4 and 5 with pairs 12, 13, 23, 24, 34, 41, 51, 52, 53 and 54. So in this tournament 1 beat 2, 1 beat 3, 2 beat 3, 2 beat 4, 3 beat 4, 4 beat 1, and 5 beat everyone.

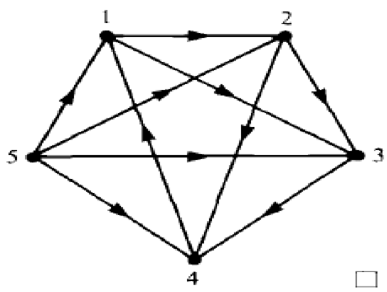


Figure 5: The graf represents a tournament of the five players.

The graphical representation of a tournament gives a special sort of 'directed graph'. A directed graph (or digraph) $(V, \bar{E})$ consists of a set of vertices, as before, and a set of edges $\bar{E}$ which is a subset of the ordered pairs $\{vw: v, w \in V\}$. A directed graph can be illustrated in the same way that we illustrated graphs, but with arrows on edges to indicate the order. The terminology of graph theory extends naturally to directed graphs: for example a 'directed path' is merely a path followed in the directions of the arrows, and the 'out-degree'/'in-degree' of a vertex is the number of edges which start/end at that vertex, etc. In the graphical representation of a tournament a directed path $p_1, p_2. \ldots, p_r$

is a sequence of players in which p_1 beat p_2, p_2 beat p_3, ... and p_{r-1}beat p_r. For instance, in the above example there is a directed path 5, 4, 1, 2, 3 which uses all the vertices. Imitating our work on Hamiltonian and semi-Hamiltonian graphs in chapter 6 we now show that there always exists a directed path in a tournament which uses each player exactly once: then in the exercises we find conditions for there to exist a directed Hamiltonian cycle (Wilson & Robin, 1996).

In a tournament of *n* players they can be labelled $p_1, p_2. ..., p_n$ so that p_1 beat p_2, p_2 beat p_3, ... and p_{n-1} beat p_n . The case *n* = 2 (or *n* = 1) being trivial. So assume that $n > 2$ and that the result is known for tournaments of fewer than *n* players. For the moment forget one of the players, p say, and consider the games between the other $n-1$ players. These will form a tournament and therefore, by the induction hypothesis, the remaining $n-1$ players can be labelled $p_1, p_2. ..., p_{n-1}$ so that

$$p_1 \text{ beat } p_2, p_2 \text{ beat } p_3, ..., p_{n-2} \text{ beat } p_{n-1} .$$

Now think about the results of *p's* games. If *p* was beaten by all the other players then we will have

$$p_1 \text{ beat } p_2, p_2 \text{ beat } p_3, ..., p_{n-2} \text{ beat } p_{n-1,}\ p_{n-1,} \text{ beat } p.$$

which is of the required form involving all *n* players in the tournament. If on the other hand *p* beat at least one of the other players, then let *i* be the lowest for which *p* beat p_i. Then if *i* = 1 we have

$$p \text{ beat } p_1, p_1 \text{beat } p_2, p_2 \text{ beat } p_3, ..., p_{n-2} \text{ beat } p_{n-1} .$$

and if *i* > 1 we have

$$p_1 \text{ beat } p_2, ..., \underbrace{p_{i-1} \text{ beat } p}_{\substack{since\ p_i\ was\ the \\ first\ beaten\ by\ p}} , p \text{ beat } p_i, ... p_{n-2} \text{ beat } p_{n-1} .$$

In both cases we have the required ordering of all the players.

A lot of results for graphs extend naturally to directed graphs: for example a directed graph has a directed Eulerian path if and only if it is `strongly connected' (in the sense that there is a directed path from any one vertex to any other) and the out-degree of each vertex equals its in-degree (Wilson & Robin, 1996).

The main result about tournaments which is going to concern us here is about the possible collection of `scores'. In a tournament of the *n* players 1, 2, ..., *n* let b_1 be the number of players beaten by player *i*: then $b_1, b_2. ..., b_n$ are the scores of the tournament (and the ordered list of them is the score vector).

Example Which of the following are possible scores in a tournament of six players?
(i) 4, 4, 4, 2, 1, 1;
(ii) 5, 3, 3, 2, 1, 1;
(iii) 5, 4, 4, 1, 1, 0.

Solution (i) Clearly these cannot be the scores in a tournament of six players because

$$4 + 4 + 4 + 2 + 1 + 1 = 16 \neq \binom{6}{2}.$$

(iii) Here (in graphical form) is a tournament with the given scores. Note that the scores correspond to the out-degrees in the graphical representation.

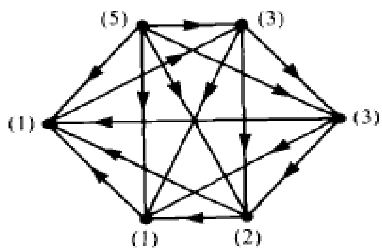


Figure 6: Graph of a tournament with the given scores.

(iv) Although the six numbers 5, 4, 4, 1, 1, 0 do add to $\binom{6}{2}$ they cannot be the scores in a tournament of six players. One way of seeing this is to consider the games between the three players who are supposed to have scores of 1, 1 and 0. Those three players will have played $\binom{3}{2} = 3$ games amongst themselves and so, no matter how bad they are, those 3 wins must have been shared between them. Therefore it is impossible for their scores to add up to less than 3.

4. Conclution

The property used in the examples turns out that any tournament score of n players, i.e. scores will add up to $\binom{n}{2}$, and any score of r will add up to at least $\binom{n}{2}$, because those *r* players will play a total of the game was only between themselves. Amazingly the converse also turns out to be true; i.e. if the set of *n* integers has both properties then it is a tournament score. This result agrees with Landau's theorem and is a convenient consequence of the harem version of Hall's theorem.

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