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Variant of Trapezoidal-Newton Method for Solving Nonlinear Equations and its Dynamics

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Abstract

This article introduces a novel approach resulting from the adaptation of Trapezoidal-Newton method variants. The iterative process is enhanced through the incorporation of a numerical integral strategy derived from two-partition Trapezoidal method. Through rigorous error analysis, the study establishes a third order convergence for this method. It emerges as a viable alternative for solving nonlinear equations, a conclusion substantiated by computational costs conducted on diverse nonlinear equation forms. Furthermore, an exploration of basin of attraction analyses that this method exhibits faster convergence compared to other Newton-type methods, albeit with a slightly expanded divergent region with a variant of Newton Simpson's method.

Keywords: Basins of attraction, nonlinear equation, Trapezoidal-Newton variant methods, Newton-Simpson variant methods

1. Introduction

The problem of finding a solution to the nonlinear equation f(x)=0. is an interesting problem that is often found in science and engineering. Therefore, the development of iterative methods used to solve problems is also growing. The development aspect of this iteration method is usually done to find faster convergent methods with short CPU time while simultaneously maximizing efficiency levels.

Another dynamic aspect that is undergoing considerable development involves the analysis of iteration methods through their dynamics. An insightful approach to understanding the behavior of these methods is by examining their basins of attraction. A basin attraction is defined as a collection of points of a dynamic system that spontaneously moves towards a particular attractor. For instance, consider basin attraction of Newton's method. By considering the root as its attractor then the dynamic convergence behavior of this method becomes discernible, particularly for specific initial values. Newton's method, as defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2 \ldots \tag{1}$$

Where $f'(x_n) \neq 0$, employs an initial value to iteratively converge towards the root of a nonlinear equation. To ensure that this method converges quadratically to the expected root, choose an initial value that is close enough to the desired root. For instance, consider the nonlinear root-finding task for the function f(z)=z^4-1, which has two real roots and two complex roots. The choice of an appropriate initial value is crucial for obtaining the desired roots. In figure 1, different-colored regions (blue, red, yellow, and green) represent the selection of initial values, each corresponding to one of the four possible roots. Consequently, an initial value chosen in the blue region, for example, yields one of the four roots of the function.

In its development, the researchers have successfully modified Newton method where one of the objectives is to increase the speed of iteration. Among these modifications are the approaches presented by (Weerakoon & Fernando, 2000), (Hansonov et al., 2002), (Chen et al., 2018) etc. Notably, these articles share a common theme of developing iterative methods though the numerical application of the Trapezoidal and Simpson methods. Weerakoon, for instance, incorporates the one-partition integral approach of the trapezoid to formulate the Newton-Trapezoidal Method (MNT). On the other hand, Hansanov and Jen Yuan Chen utilize the Simpson 1/3 and 3/8 integrals, respectively, leading to the development of Newton-Simpson 1/3 method (MNS3) and Newton-Simpson 3/8 method

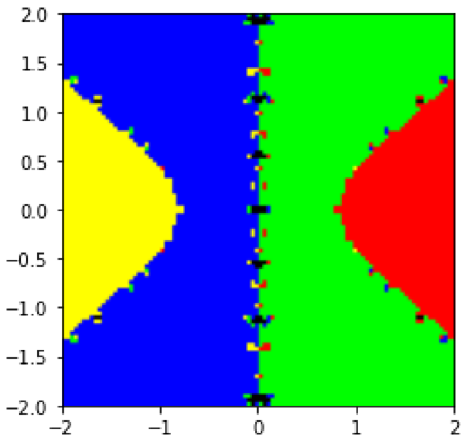
(MNS8). These modifications represent innovative strategies at optimizing the Newton method for improved performance in terms of speed and efficiency during iterative process.

In this paper, our focus lies in undertaking a thorough investigation into the advancement of root-finding methods, specifically leveraging the concept of the two-partition trapezoid. The derived method will be analyzed and tested to solve several nonlinear equations in research methodology section. Additionally, an examination of the basins of attraction for each method will be conducted in results section. Through the analysis of computational test outcomes and dynamic comparisons, these approaches are expected to provide a deeper insight into the behavior and performance of the proposed method. Furthermore, this study aims to offer novel perspectives on the subject, contributing to a broader understanding of root-finding method development.

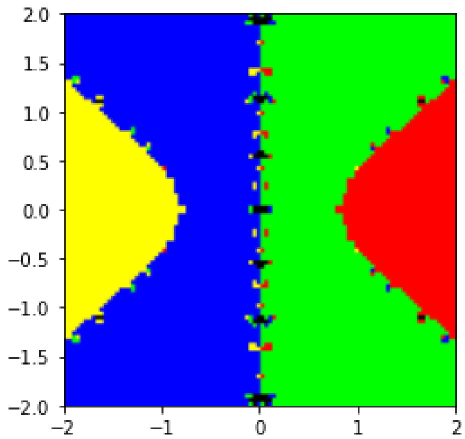
2. Research Methodology

2.1. Basins of Attraction

(Cayley, 1879) introduced the concept of basins of attraction as a technique to illustrate the impact of various initial values on the behavior of a function. This approach facilitates the comparison of different iteration methods based on the convergence region within the attraction basins. In this context, the effectiveness of iterative methods is often characterized by the breadth of their convergence area. The width of the convergence area refers to the quantity of convergent points toward the root of $f(x) = 0$ within the specified range. The comparison between method A and method B, as depicted in the accompanying image, demonstrates that method B is superior to method B due to its broader convergence area. This assessment is supported by the observation and quantification of the number of convergent points, indicated by the black spots, in both images.



a) Method A



b) Method B

Figure 1: Illustration on comparison of basins of attraction of two different methods of a function.

(Stewart, 2001) employed the concept of basins of attraction to conduct a comparative analysis between Newton' iterative method and the iteration method by (Halley, 1800). In instances involving a double root of a known multiplicity in a nonlinear equation $f(x) = 0$, numerous researchers have explored and compared various methods with different convergence orders by examining their basins of attraction. Notable examples include (Neta et al., 2012) who presented basins of attraction for methods exhibiting a three-order convergence.

Similarly, (Solaiman & Hashim, 2021) and (Putra et al., 2022) undertook a comparison between various forms of optimal and non-optimal iteration methods, particularly those with a sixteenth conventional order, by evaluating their basins of attraction. These studied demonstrates that an optimal iterative method may not always be the most effective choice for nonlinear equations, emphasizing the importance of considering specific examples.

Numerous researchers have contributed to the field by proposing iterative methods for solving nonlinear equations and presenting their associated attraction basins. Notable mentions include (Behl et al., 2019) and (Solaiman & Hashim, 2021).

3. Results and Discussion

3.1. Variant of Trapezoidal-Newton Method

In 1998, (Weerakoon & Fernando, 2000) introduced a variant of Newton method that was developed with the idea of integration of Newton's method.

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\mu)d\mu \quad (2)$$

Employing the trapezoid approximation to calculate the integral area of this equation (2) then we obtain a new method with the form of iteration

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, n = 0,1,2 \ldots \quad (3)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$. Weerakoon has designated this method as a variation of the Newton-Trapezoidal method (MNT). Notably, MNT exhibits a third-order convergence and mandates three function evaluations per iteration. Consequently, the efficiency index for this method is calculated at 1.442, surpassing that of the Newton method. This underscores the enhanced efficiency of MNT in comparison to the traditional Newton method.

(Hansonov et al., 2002) introduced a modification to Newton's method, employing the Simpson approximation 1/3 to compute the integral area in equation (2). This resulted in the creation of the Newton-Simpson method variant 1/3 (MNS3).

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)+f'(y_n)+4f'(z_n)}, n = 0,1,2 \ldots \quad (4)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ and $z_n = x_n - \frac{f(x_n)}{2f'(x_n)}$. This method has the same three-order convergence as MNT. However, it is less efficient than MNT because it requires five function evaluations on each iteration so that the efficiency index of this method is 1,246 which is smaller than MTN even Newton. A new variation of the Newton method was developed by (Chen et al., 2018) employing the Simpson 3/8 approximation to calculate the integral area on the equation (2) thus obtaining a Newton-Simpson variant of the 3/8 method (MNS8)

$$x_{n+1} = x_n - \frac{8f(x_n)}{f'(x_n)+3f'\left(x_n-\frac{1}{3}h(x_n)\right)+3f'\left(x_n-\frac{2}{3}h(x_n)\right)+f'(x_n-h(x_n))}, \; n = 0,1,2,\ldots \quad (5)$$

where $h(x_n) = \frac{f(x_n)}{f'(x_n)}$. The MNS8 method is also less efficient because it requires five function evaluations on each iteration so the efficiency index of this method is 1.246 which is equal to MNS3 but smaller than MTN.

Integral form of (2) is approximated by employing interval $[x_n, x_{n+1}]$ that is divided into two partitions. Hence

$$\int_{x_n}^{x} f'(\mu)d\mu \approx \frac{x-x_n}{4}\left(f'(x_n) + 2f'\left(\frac{x_n+x_{n+1}}{2}\right) + f'(x_{n+1})\right) \quad (6)$$

Substituting (6) into (2) we obtain

$$f(x) \approx f(x_n) + \frac{x-x_n}{4}\left(f'(x_n) + 2f'\left(\frac{x_n+x_{n+1}}{2}\right) + f'(x_{n+1})\right) \quad (7)$$

Since $f(x) = 0$, then we have

$$f(x_n) + \frac{x-x_n}{4}\left(f'(x_n) + 2f'\left(\frac{x_n+x_{n+1}}{2}\right) + f'(x_{n+1})\right) \approx 0 \quad (8)$$

Solving equation (8) in *x* gives,

$$x \approx x_n - \frac{2f(x_n)}{\frac{f'(x_n)+f'(x_{n+1})}{2}+f'\left(\frac{x_n+x_{n+1}}{2}\right)} \quad (9)$$

Let *x* be the-$n+1$ approximation root, then (9) becomes

$$x_{n+1} = x_n - \frac{2f(x_n)}{\frac{f'(x_n)+f'(y_n)}{2}+f'\left(\frac{x_n+y_n}{2}\right)} \quad (10)$$

with $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$. This equation will be referred to as variant of Trapezoidal-Newton two partition method or MNT2. The error analysis of the method will be presented in the following section.

3.2. Error Analysis

Theorem 1. Assume $f: D \to \mathbb{R}$ for open interval D. Assume that $f \in C^3(D)$. In addition, let α be the simple root of the nonlinear equation $f(x) = 0$, and the initial value of x_0 is taken sufficiently close to α, then the Trapezoidal-Newton method of two partitions (MNT2) has an order of cubic convergence.

Proof. Let α be the simple root of $f(x) = 0$. This means that $f(\alpha) = 0$ and $f(\alpha) \neq 0$. Assume $e_n = x_n - \alpha$. Employing Taylor expansion up to order three to $f(x)$ around α, we obtain,

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \frac{f'''(\alpha)}{3!}(x-\alpha)^3 + O((x-\alpha)^4) \quad (11)$$

Upon evaluating $x = x_n$ and enforcing the above assumptions, equation (11) becomes

$$f(x_n) = f'(\alpha)e_n + \frac{f''(\alpha)}{2!}e_n^2 + \frac{f'''(\alpha)}{3!}e_n^3 + O(e_n^4)$$

$$f(x_n) = f'(\alpha)\left[e_n + \frac{f''(\alpha)}{2!\,f'(\alpha)}e_n^2 + \frac{f'''(\alpha)}{3!\,f'(\alpha)}e_n^3 + O(e_n^4)\right]$$

$$f(x_n) = f'(\alpha)[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)] \tag{12}$$

where $C_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)},\ j = 2{,}3$. Utilizing the same technic, we obtain the Taylor expansion of $f'(x)$ around α as follows

$$f'(x_n) = f'(\alpha)[1 + 2C_2e_n^1 + 3C_3e_n^2 + O(e_n^3)] \tag{13}$$

From (12) and (13) and geometric series, one attains

$$\frac{f(x_n)}{f'(x_n)} = e_n - C_2e_n^2 + (2C_2^2 - 2C_3)e_n^3 + O(e_n^4). \tag{14}$$

Next, y_n and $f'(y_n)$ are obtained using the same technic

$$y_n = \alpha + C_2e_n^2 + (2C_2^2 - 2C_3)e_n^3 + O(e_n^4) \tag{15}$$

and

$$f'(y_n) = f'(\alpha)[1 + 2C_2^2e_n^2 + (4C_2C_3 - 4C_2^3)e_n^3 + O(e_n^4)]. \tag{16}$$

Substituting x_n and y_n into (15), yields

$$f'\left(\frac{x_n+x_n}{2}\right) = f'(\alpha)\left[1 + C_2e_n + \left(C_2^2 + \frac{3}{4}C_3\right)e_n^2 + \left(\frac{7}{2}C_2C_3 - C_2^3 + \frac{1}{2}C_3\right)e_n^3 + O(e_n^4)\right] \tag{17}$$

Furhtermore, (12), (16) and (17) and simplifying we obtain

$$\frac{2f(x_n)}{\frac{f'(x_n)+f'(y_n)}{2} + f'\left(\frac{x_n+y_n}{2}\right)} = e_n - \left(C_2^2 + \frac{1}{8}C_3\right)e_n^3 + O(e_n^4)$$

It shows that the error equation of MNT2 is

$$e_{n+1} = \left(C_2^2 + \frac{1}{8}C_3\right)e_n^3 + O(e_n^4) \tag{18}$$

Hence, it can be concluded that MNT2 has third order of convergence. □

3.3. Computational Tests

In this section, a computational test is done by comparing number of iterations (*n*) and error produced during the computations from several iteration methods of third order. The computation is done for the following methods:

- **MN**: Newton's method,
- **MNT**: Newton-Trapezoidal method,
- **HNS3**: Newton-Simpson 1/3 method
- **HNS8**: Newton-Simpson 3/8 method
- **MNT2**: Trapezoidal-Newton two partition method

Nonlinear equations that are involved in the simulation are taken from (Weerakoon and Fernando, 2000):

- $f_1(x) = x^3 + 4x^2 - 10$
- $f_2(x) = sin^2(x) - x^2 + 1$
- $f_3(x) = x^2 - e^x - 3x + 2$
- $f_4(x) = xe^{x^2} - sin^2(x) + 3\cos(x) + 5$

Stopping criteria for the simulations are:

- $|\alpha - x_n| < tolerance$
- Maximum iterations exceeded

In this simulation, we used tolerance of 1×10^{-50} and limit number of iterations up to 100. The results can be in these possibilities: Number of iterations exists if the method converges, div if the method converges, and NA if the method is not applicable.

Numerical computations of the compared methods are presented in Table 1 and Table 2.

Table 1: Comparison of number of iterations for several methods with different initial guesses

Non-linear functions	x_0	Number of Iterations				
		MN	**MNT**	**MNS3**	**MNS8**	**MNT2**
$f_1(x)=0$	1.0*	7	4	4	4	4
	2.0	7	5	5	5	5
	4.0	8	6	6	6	6
$f_2(x)=0$	1.2*	7	4	4	4	4
	2.0	7	5	5	5	5
	4.0	8	5	5	5	5
$f_3(x)=0$	0.0*	6	4	4	4	4
	1.0	6	4	4	4	4
	2.0	6	5	4	4	5
$f_4(x)=0$	-1.0*	5	4	4	4	4
	-0.5	10	9	7	7	8
	-3.0	17	11	11	11	11

Table 2: Comparisons of approximation error for the first three iteration

Non-linear functions	x_0	Approximation Error				
		MN	**MNT**	**MNS3**	**MNS8**	**MNT2**
$f_1(x)=0$	1.0*	3.664e-03	2.285e-06	1.366e-06	1.366e-06	1.564e-06
	2.0	8.071e-03	1.196e-05	8.049e-06	8.049e-06	8.924e-06
	4.0	3.069e-01	3.892e-02	3.083e-02	3.083e-02	3.276e-02
$f_2(x)=0$	1.2*	1.436e-03	4.458e-07	3.025e-07	3.025e-07	3.346e-07
	2.0	1.248e-02	3.168e-05	3.956e-05	3.954e-05	3.754e-05
	4.0	1.105e-01	5.530e-04	5.008e-03	4.683e-03	3.457e-03
$f_3(x)=0$	0.0*	5.340e-06	7.802e-12	7.319e-14	7.313e-14	4.323e-13
	1.0	1.202e-05	1.689e-07	2.900e-13	3.512e-13	9.862e-10
	2.0	3.309e-03	7.591e-04	1.066e-05	1.053e-05	5.498e-05
$f_4(x)=0$	-1.0*	1.904e-08	3.230e-17	9.416e-18	9.416e-18	1.330e-17
	-0.5	1.997e-01	7.260e-02	1.787e-01	1.804e-01	1.473e-01
	-3.0	1.743e-01	2.699e-01	2.988e-01	2.989e-01	2.910e-01

Observing Table 1, it can be asserted that, for initial values in close proximity to the root, all methods effectively approximate the anticipated root. Similar to Newton's method, when initial values are relatively close to the root (*), these methods exhibit fewer iterations.

Upon examining the number iterations required to locate the expected side root, it becomes evident that the Newton method (MN) necessitates more iterations compared to alternative methods. This outcome aligns with expectations, considering Newton's lower convergence order relative to other methods.

The comparison of the same method with MNT2 indicates a comparable performance to other variants of the Newton method. This consistency is also reflected in the comparison of error values, as presented in Table 2.

3.4. Analysis on The Basins of Attraction

In this section we will conduct a dynamic study of six iteration methods with combinations of averages to solve nonlinear equations $g(z) = 0$, where functions $g: \mathbb{C} \to \mathbb{C}$ are in complex fields.

In the following section will be given two different polynomial examples of different orders. In addition, we will construct a basin image of attraction from the six iteration methods to be compared. The method is to take a square field $[-2.2] \times [-2.2] \subset \mathbb{C}$ divided into 100 partitions which means there will be 10,000 cells. Then assign a different color to different roots, which means for each $z_0 \in D \subset \mathbb{C}$ is given a color that has been specified when iterations converge and given a black color when the initial values are not convergent to one of the existing roots. This procedure starts with taking the initial value z_0 with error tolerance of 10^{-10} and maximum iteration of 100. In addition to generating a basin image of attraction, each program will also generate a lot of non-convergence points and long computational time (CPU time) required to obtain a solution. The recorded CPU time is the average CPU time of 100 attempts. In this case we use Python for computational purposes to solve two complex polynomial examples used.

Example 1. Given a nonlinear equation

$$g_1(z) = z^4 - \frac{5}{4}z^2 + \frac{1}{4}.$$

This polynomial has four different real roots namely $\left\{1, -\frac{1}{2}, \frac{1}{2}, -1\right\}$. The basins of attraction for each comparing method of this polynomial is given in Figure 4.1 while many non-convergence initial values (black) and CPU time are given in Table 3.

Tabel 3: Comparisons of number of divergent points and computation time of studied methods in solving $g_1(z) = 0$ in complex plane

Methods	Divergent points number	Divergent points %	Computation time (second)
MNT	64	0.64	0.50489
MNS3	20	0.20	0.58329
MNS8	20	0.20	0.47169
MNT2	28	0.28	0.44648

Figure 4 represents the convergence area of four real roots of the nonlinear equation $g_1(z) = 0$. Comparison of methods from Table 3 shows that variants of Newton Simpson's method have narrower divergence areas (20 points). However, the proposed MNT2 method has a slightly divergent point (28 points) compared to the MNT method, whereas when compared with its computational time, the method has the fastest computing time than other variants of Newton's method.

Example 2. Given the nonlinear equation

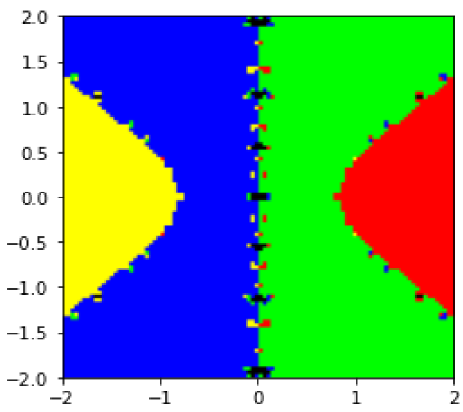
$$g_2(z) = z^4 - 1.$$

This polynomial has four roots consisting of two real roots and two complex roots namely $\{1, -1, i, -i\}$. A basin image of attraction for each comparable method of this polynomial is given in Figure 5 while many non-convergence initial values (black) and CPU time are given in Table 4.

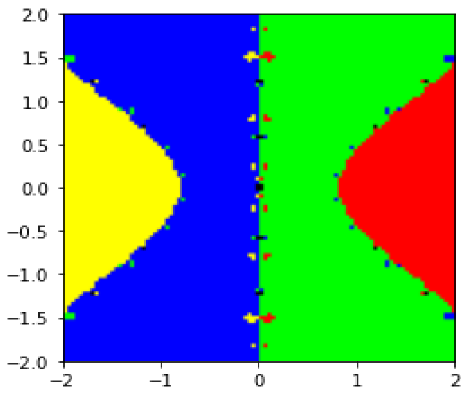
Table 4: Comparisons of number of divergent points and computation time of studied methods in solving $g_2(z) = 0$ in complex plane

Methods	Divergent points number	Divergent points %	Computation time (second)
MNT	2528	25.28	0.51465
MNS3	1578	15.78	0.60459
MNS8	1580	15.80	0.58897
MNT2	1744	17.44	0.49329

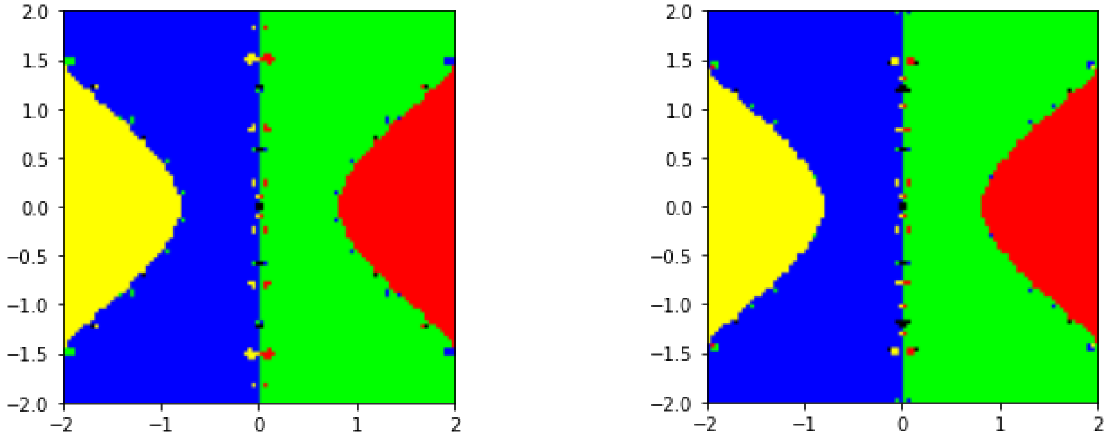
Figure 5 illustrates the convergence area for four roots, comprising two real and two complex solutions of the nonlinear equations $g_2(z) = 0$. In coherence with the findings in Table 4, this comparative analysis highlights that the Newton-Simpson method variants exhibit a narrower divergence area, with 1578 points for MNS3 and 1580 points for MNS8. Conversely, the proposed MNT2 method displays a slightly larger number of divergence points (1744 points) when compared to the MNT method. Notably, despite the increased divergence points, the MNT2 method demonstrates the fastest computation time compared to other variants of the Newton method.



a) MNT

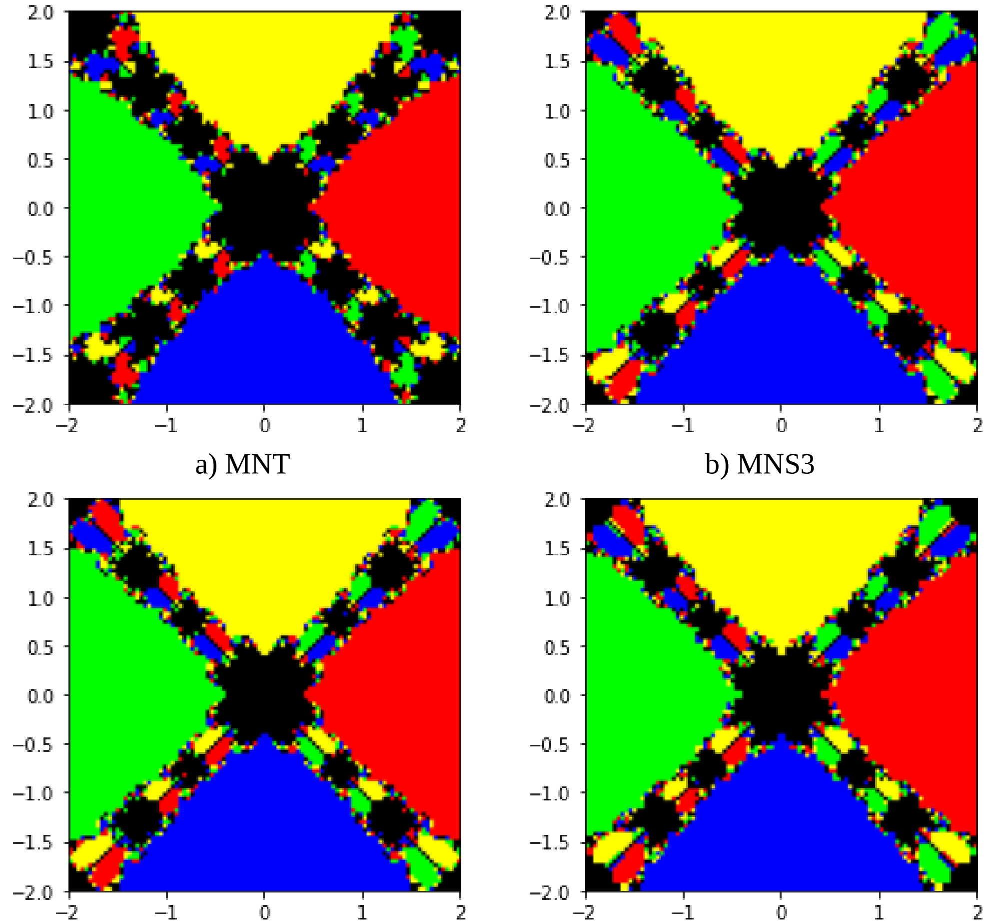


b) MNS3



c) MNS8 d) MNT2

Figure 4: Basin of attraction metode iterasi untuk $g_1(z) = z^4 - \frac{5}{4}z^2 + \frac{1}{4}$



a) MNT b) MNS3

c) MNS8 d) MNT2

Figure 5: Basin of attraction iteration methods for $g_2(z) = z^4 - 1$

4. Conclussion

This study introduces a novel adaptation of Newton's method, incorporating a two-partition Trapezoid approach. The convergence analysis reveals that this method exhibits a third-order convergence, positioning it as a viable alternative for iterative solutions to nonlinear equations.

An examination of the basins of attraction indicates that this newly proposed method achieves faster convergence to other Newton metho variants under consideration. However, it noteworthy that it also encompasses a slightly larger divergent area when contrasted with variants of Newton's method developed with the Simpson's approach.

Additionally, the study concludes that iteration methods with identical convergence orders and the same number of function evaluations yield different basins of attraction. This observation suggests that, when coupled with the efficiency index of an iteration method, the indexed efficiency alone may not suffice for making comparison between iterations with matching convergence orders and function evaluation numbers

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