Available online at https://journal.rescollacomm.com/index.php/ijqrm/index

International Journal of Quantitative Research and Modeling

e-ISSN 2721-477X
p-ISSN 2722-5046

Vol. 5, No. 3, pp. 323-330, 2024

Application of the Leslie Matrix on Female Birth Rates and Life Expectancy in the Special Region of Yogyakarta

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Abstract

This study aims to predict the number and growth rate of the female population in the Special Region of Yogyakarta for 2025 using the Leslie Matrix model. The matrix utilizes fertility rates and female life expectancy across different age intervals. The data used includes the female population from 2015 and 2020, alongside Age-Specific Fertility Rate (ASFR) data for the same period. By applying the dominant eigenvalue of the Leslie matrix, the study finds that the growth rate of the female population in Yogyakarta is projected to increase, with a dominant eigenvalue of 1.252. The female population is predicted to reach 2,409,852 by 2025, an increase from 1,983,800 in 2020. These findings are expected to inform population management and development planning in Yogyakarta.

Keywords: leslie matrix, female population, population growth rate, eigenvalue, fertility rate, life expectancy.

1. Introduction

The Special Region of Yogyakarta, with its diversity, represents a miniature version of Indonesia. In terms of population size, it is not very large, but in terms of population quality, it has excellent potential. Yogyakarta's Human Development Index in 2022 was classified as very high. With the right strategies, the population, as a potential resource, can become a national strength in realizing Indonesia's vision of progress (BPS Yogyakarta Province).

Changes in population size are influenced by internal factors within the population, including birth, death, and survival rates. Changes in population size are referred to as population growth. Population growth provides information on whether the population size in the following year will increase, decrease, or remain stable (Pratama et al., 2013).

One of the models used to determine population growth is the Leslie model, which utilizes a mathematical approach, specifically matrices. In the Leslie model, birth and death processes depend on age and play an important role in population growth. Generally, the growth of living organisms is a continuous process. However, population studies also need to be approached from a discrete time perspective. The use of discrete patterns is also based on population observations, which are generally conducted at specific time intervals, such as daily, weekly, or over other units of time depending on the research design. Based on these considerations, the development of growth models, in addition to continuous solutions, also needs to be evaluated more thoroughly using discrete methods (Maryati et al., 2021).

Several studies have been conducted on the Leslie Matrix, including a study by Maryati et al. (2021), which used the Leslie matrix to predict the number and growth rate in West Java in 2021, where the female population tended to increase. Furthermore, in a previous study, the Leslie matrix was used to find the dominant eigenvalue, with several factors influencing population growth, such as fertility, survival rates, and age distribution (Anggreini & Hastari, 2017).

Based on the above explanation, research is needed to determine the growth rate of the female population and to predict the female population size in 2025 in Yogyakarta, based on birth rates and life expectancy using the eigenvalues and eigenvectors of the Leslie matrix.

2. Literature Review

2.1. Eigenvalue

Eigenvalue (λ) is a characteristic value of a matrix, representing the matrix in the form of multiplication with a vector. Several sets of simultaneous equations can be expressed in matrix form:

$$[A]\{X\} = \lambda \{X\} \tag{1}$$

where A is a square matrix, X is a vector, and λ is a scalar. From equation (1), it is evident that the solution to the equation is a vector $\{X\}$ that, when multiplied by the matrix $[A]$, yields the same result as multiplying the scalar value λ by $\{X\}$. This equation can be written in homogeneous form as:

$$([A] - \lambda[I])\{X\} = 0 \tag{2}$$

where $[I]$ is the identity matrix with the same order as matrix $[A]$. Since the above equation is homogeneous, it is consistent and always has a trivial solution ($X = 0$). Non-trivial solutions only exist if the determinant of the coefficient matrix equals zero.

$$|A - \lambda I| = \begin{vmatrix} A_{11} - \lambda & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} - \lambda \end{vmatrix} = 0 \tag{3}$$

The scalar value λ, for which non-trivial solutions exist, is called the eigenvalue.

2.2. Leslie Matrix

The Leslie matrix is constructed to estimate the number and growth rate of a population. The Leslie matrix was introduced by P. H. Leslie in 1945, an ecology expert. The Leslie matrix is a square matrix where the entries in the first row represent female fertility rates, the sub-diagonal consists of female survival rates, and all other entries besides the first row and sub-diagonal are zeros. It is assumed that there is no migration within the population (Pratama et al., 2013). For simplicity, it is assumed that age groups are uniform over a specific period, and only the female population is used in the Leslie matrix calculation (Leslie, 1948). In another article, it was also mentioned that since only females can give birth, the Leslie matrix calculations focus solely on the female population. Female fertility and survival rates are factors that influence population estimates for future years using the Leslie matrix (Sanusi, Sukarna, and Ridiawati, 2019)

Let a_l represent the fertility rate of females in the age group l, and b_l represent the survival rate of females in age group l, which is the probability of females surviving from age group l to age group $l + 1$ at time t. The values of a_l and b_l can be determined by:

$$a_l = \frac{number\ of\ births\ in\ age\ group{-}l}{n_l(t)} \tag{4}$$

$$a_1 \geq 0 \text{ for } l = 1,2, \dots, n$$

$$b_l = \frac{n_{l+1}(t)}{n_l(t-1)} \tag{5}$$

The general form of the Leslie matrix is (Marzuki et al., 2016):

$$L = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \end{bmatrix} \tag{6}$$

Assume there is at least one age group a_l such that $a_l > 0$, because if $a_l = 0, \forall_l$, it means there are no births in that age group. The age group a_l with a value greater than zero is called the fertility age group. Similarly, $b_l \neq 0$ because if $b_l = 0$, it means that no females survive to the next age group.
In the Leslie matrix, females are classified into several age groups with equal age ranges. Let M represent the maximum age limit for females in a population, and the female population is divided into l age groups, so each age group has an age range of M/l years. The determination of female age groups can be seen in Table 1.

Table 1: Determination of Age Groups

Age Group	Age Range	Age Group	Age Range
1	$[0, M/l]$	$\vdots$	$\vdots$
2	$[M/l, 2M/l)$	$l-1$	$[(l-2)M/l, (l-1)M/l]$

Assuming the female population for each age group is known at time t, let $n_1(t)$ represent the female population in the first age group at time t, $n_2(t)$ represent the female population in the second age group at time t, and so on, until the female population in the l-th age group is denoted by $n_l(t)$, so the total population at time t is:

$$n(t) = n_1(t) + n_2(t) + n_3(t) + \cdots + n_l(t) \tag{7}$$

The total female population at each age group at time t is written as:

$$n(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \end{bmatrix} \tag{8}$$

where the vector $n(t)$ is called the initial age distribution vector. At time $t + 1$, let $n_1(t+1)$ denote the female population in the first age group, $n_2(t+1)$ denote the female population in the second age group, and so on, until $n_l(t+1)$, representing the female population in the l-th age group, so the total population at time $t + 1$ is:

$$n(t+1) = n_1(t+1) + n_2(t+1) + n_3(t+1) + \cdots + n_l(t+1) \tag{9}$$

The total female population at each age group at time $t+1$ is written as:

$$n(t) = \begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+1) \\ n_4(t+1) \end{bmatrix} \tag{10}$$

The female population in the first age group at time $t + 1$ is defined as:

$$n_1(t+1) = a_1 n_1(t) + a_2 n_2(t) + a_3 n_3(t) + \cdots + a_l n_l(t), \tag{11}$$

The number of female births occurring from time t to $t + 1$ represents the female population in the first age group at time $t + 1$. Next, the average number of females in age group l at time t who survive to age group $l + 1$ at time $t + 1$ is defined as the total female population in age group $l + 1$ at time $t + 1$, for $l = 1, 2, 3, \ldots, n - 1$. Mathematically, this is expressed as:

$$n_{l+1}(t+1) = b_l n_l(t), l = 1,2,3, \ldots, l-1 \tag{12}$$

Thus, the female population growth model becomes:

$$n_1(t+1) = a_1 n_1(t) + a_2 n_2(t) + a_3 n_3(t) + \cdots + a_l n_l(t)$$

And

$$n_{l+1}(t+1) = b_l n_l(t), l = 1,2,3, \ldots, l-1.$$

This can be written as:

$$n(t+1) = \begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+1) \\ \vdots \\ \vdots \\ l(t+1) \end{bmatrix} = \begin{bmatrix} a_1 a_2 \dots a_{n-1} a_l \\ b_1 \; 0 \cdots \; 0 \; 0 \\ 0 \; b_2 \ddots \; 0 \; 0 \\ \vdots \; \vdots \; \ddots \; \vdots \; 0 \\ 0 \; 0 \cdots b_{n-1} \; 0 \end{bmatrix} \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \end{bmatrix} \tag{13}$$

or the population growth model can be rewritten as:

$$n(t+1) = L \times n(t) \tag{14}$$

with the following explanation:

$n(t + 1)$: Population vector containing the estimated female population at time $t+1$.
L : Leslie matrix of size $n \times n$.
$n(t)$: Population vector containing the population by age group at time t

2.3. Eigenvalue of the Leslie Matrix

The population growth rate can be determined by finding the dominant eigenvalue of matrix L. The following theorems are needed to determine the dominant eigenvalue of a Leslie matrix (Anggreini and Hastari, 2017):

Theorem 1
A Leslie matrix L has a unique positive eigenvalue λ_i. This eigenvalue has multiplicity 1 and is associated with an eigenvector $\mathbf{x}_1$ whose entries are all positive.

Theorem 2
If λ_i is the unique positive eigenvalue of a Leslie matrix L and λ_i is any real or complex eigenvalue of L, then $|\lambda_i| \leq \lambda_1$.

Theorem 3
If two consecutive entries a_l and a_{l+1} in the first row of a Leslie matrix L are non-zero, then the positive eigenvalue of L is dominant. An eigenvalue is considered dominant if $|\lambda_l| < \lambda_1$.
Three scenarios will arise based on the dominant eigenvalue (λ_1):

- The population will eventually tend to increase if $\lambda_1 > 1$,
- The population will eventually tend to decrease if $\lambda_1 < 1$,
- The population will eventually remain stable if $\lambda_1 = 1$

3. Methods

This research falls under the category of applied research, where the Leslie Matrix model is applied to predict the female population in the Special Region of Yogyakarta in 2015 and the population growth rate in Yogyakarta. The data was obtained from the websites https://www.bps.go.id and https://dalduk.jogjaprov.go.id. The data used includes the female population in Yogyakarta in 2015 and 2020, based on age range, and the Age-Specific Fertility Rate (ASFR) for Yogyakarta from 2015 to 2020.
The steps taken in this research are as follows:

(a) Determine the number of births in 2015-2020. This value is obtained using the formula:

$$\text{number of births} = \frac{ASFR \times number\ of\ women}{1000}$$

which is calculated for each age range.

(b) Determine the female fertility rate (a_i) and female survival rate (b_i).
(c) Insert the values of (a_i). and (b_i). into the Leslie matrix.
(d) Calculate the dominant eigenvalue to determine the female population growth rate.
(e) Multiply the Leslie matrix by the female population data from 2020 to predict the female population in 2025.

4. Results and Discussion

To predict the female population in 2025 $(t = 3)$, data from the female population in 2015 $(t = 1)$and 2020 $(t = 2)$ is required. The data on the female population in 2015 and 2020 by age range is presented in Table 2.

Table 2: Female Population in Yogyakarta in 2015 and 2020

Class	Age	2015	2020
1	0 - 4	123417	136293
2	5 - 9	124273	127685
3	10 - 14	130871	126267
4	15 - 19	139003	139731
5	20 - 24	143771	161274
6	25 - 29	144529	159109
7	30 - 34	144678	147777
8	35 - 39	139988	147351
9	40 - 44	136174	140478
10	45 - 49	127140	135583
11	50 - 54	117653	125981
12	55 - 59	100792	115484
13	60 - 64	83151	98264
14	65 - 69	67826	78496
15	70 - 74	53808	59666
16	75+	76341	84361
	total	1853415	1983800

The number of births in 2015-2020 is calculated annually for each age range. For example, in 2015, the ASFR for the 15-19 age group was 33, and the number of females in this age range was 139,003. The number of births is calculated as follows:

$$\text{number of births} = \frac{ASFR \times number\ of\ woment}{1000} = \frac{33 \times 139003}{1000} = 4587.099 \approx 4588.$$

Using this method, the number of births for 2015-2020 is as follows:

Table 3: Number of Births in 2015-2020

Class	Age	2015	2016	2017	2018	2019	2020	total
1	0 - 4	0	0	0	0	0	0	0
2	5 - 9	0	0	0	0	0	0	0
3	10 - 14	0	0	0	0	0	0	0
4	15 - 19	4588	5041	4640	4224	4636	1612	24741
5	20 - 24	16102	16942	18192	19237	19568	11468	101509
6	25 - 29	18211	19041	20027	21186	21596	19744	119805
7	30 - 34	14757	14857	15215	15700	15752	14760	91042
8	35 - 39	8399	8630	9293	8531	9499	8137	52490
9	40 - 44	2996	2737	2618	2215	2785	2135	15487
10	45 - 49	381	387	523	133	268	145	1838
11	50 - 54	0	0	0	0	0	0	0
12	55 - 59	0	0	0	0	0	0	0
13	60 - 64	0	0	0	0	0	0	0
14	65 - 69	0	0	0	0	0	0	0
15	70 - 74	0	0	0	0	0	0	0
16	75+	0	0	0	0	0	0	0
	Total	65434	67635	70509	71227	74106	58002	406912

Next, the female fertility rate (a_i) and the survival rate (b_i) for each age interval from 2015 to 2020 are calculated using equations (4) and (5). For example, for the 15-19 age range:

$$a_4 = \frac{number\ of\ births\ in\ class - 4}{n_l(2015)} = \frac{24741}{139003} = 0.1780$$

This means that in the 15-19 age range, 17.8% of women in Yogyakarta will give birth to a child. Additionally, the survival rate is:

$$b_4 = \frac{n_{4+1}(2020)}{n_4(2015)} = \frac{161274}{139003} = 1.16$$

The results for ala_lal and blb_lbl for each age range are presented in Table 4.

Table 4: Female Fertility Rates (a_i) and Survival Rates (b_i)

Class	interval			a_l	b_l
1	0	-	4	0	1.034582
2	5	-	9	0	1.016045
3	10	-	14	0	1.0677
4	15	-	19	0.17799	1.16022
5	20	-	24	0.70605	1.106684
6	25	-	29	0.828931	1.022473
7	30	-	34	0.629274	1.018476
8	35	-	39	0.374961	1.0035
9	40	-	44	0.113728	0.99566
10	45	-	49	0.014458	0.990884
11	50	-	54	0	0.981564
12	55	-	59	0	0.974919
13	60	-	64	0	0.944018
14	65	-	69	0	0.879692
15	70	-	74	0	1.567815
16	75+			0	-

The values of a_l and b_l obtained are then inserted into the Leslie matrix, resulting in the following Leslie matrix:

$$L := \begin{bmatrix}
0 & 0 & 0. & 0.1780 & 0.7060 & 0.8289 & 0.6293 & 0.3750 & 0.1137 & 0.0145 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0346 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0160 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.0677 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.160 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.1067 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0224 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0185 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0035 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9957 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9909 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9816 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9749 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9440 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.880 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.568 & 0
\end{bmatrix}$$

Using MAPLE, the eigenvalues of the Leslie matrix are:

$$\lambda_1 = 1.252,$$
$$\lambda_2 = \lambda_3 = \cdots = \lambda_7 = 0,$$
$$\lambda_8 = 0.5503 + 0.937i,$$
$$\lambda_9 = 0.5503 - 0.937i,$$
$$\lambda_{10} = -0.0603 + 0.6593i,$$
$$\lambda_{11} = -0.0603 - 0.6593i,$$
$$\lambda_{12} = 0.5331 + 0.5693i,$$
$$\lambda_{13} = 0.5331 - 0.5693i,$$
$$\lambda_{14} = -0.6584,$$
$$\lambda_{15} = -0.2537 + 0.1392i,$$
$$\lambda_{16} = -0.2537 - 0.1392i.$$

From these eigenvalues, the dominant eigenvalue is $\lambda_1 = 1.252$, because $|\lambda_1| > |\lambda_l|, l = 2 \ldots 16$. Since $\lambda_1 = 1.252 > 1$, the female population in Yogyakarta in 2025 is expected to increase.
To predict the female population at time $t + 1$, we use equation (14):

$$n(t + 1) = L \times n(t)$$

The prediction for the female population in Yogyakarta in 2025($n(3)$) is obtained using the equation:

$$n(2 + 1) = L \times n(2)$$

with $n(2)$ being the female population in 2020. The predicted female population in Yogyakarta in 2025 is as follows:

Table 5: Female Population in Yogyakarta in 2015, 2020, and Prediction for 2025

Class	Age	2015	2020	Predicted 2025
1	0 - 4	123417	136293	436808
2	5 - 9	124273	127685	141009
3	10 - 14	130871	126267	129728
4	15 - 19	139003	139731	134816
5	20 - 24	143771	161274	162088
6	25 - 29	144529	159109	178482
7	30 - 34	144678	147777	162674
8	35 - 39	139988	147351	150511
9	40 - 44	136174	140478	147867
10	45 - 49	127140	135583	139874
11	50 - 54	117653	125981	134350
12	55 - 59	100792	115484	123663
13	60 - 64	83151	98264	112586
14	65 - 69	67826	78496	92762
15	70 - 74	53808	59666	69077
16	75+	76341	84361	93557
	total	1.853.415	1.983.800	2.409.852

5. Conclussion

Based on the data analysis using the Leslie Matrix with MAPLE 16, it was found that the dominant eigenvalue $\lambda_1 = 1.252$. This indicates that the growth rate of the female population in the Special Region of Yogyakarta in 2025 is expected to increase because $\lambda_1 = 1.252 > 1$. The predicted female population in Yogyakarta for 2025 is 2,409,852 people.

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