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On Generalization of Fibonacci, Lucas and Mulatu Numbers

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Abstract

Fibonacci numbers, Lucas numbers and Mulatu numbers are built in the same method. The three numbers differ in the first term, while the second term is entirely the same. The next terms are the sum of two successive terms. In this article, generalizations of Fibonacci, Lucas and Mulatu (GFLM) numbers are built which are generalizations of the three types of numbers. The Binet formula is then built for the GFLM numbers, and determines the golden ratio, silver ratio and Bronze ratio of the GFLM numbers. This article also presents generalizations of these three types of ratios, called Metallic ratios. In the last part we state the Metallic ratio in the form of continued fraction and nested radicals.

Keywords: GFLM numbers, Fibonacci numbers, Lucas numbers, Mulatu numbers, Binet formula, Golden ratio, Silver ratio, Bronze ratio, Metallic ratio, continued fraction, nested radicals.

1. Introduction

Fibonacci numbers are found by Leonardo Fibonacci. Fibonacci was born in Pisa in 1170 (Koshy, 2001; Kalman, and Mena, 2003; Falcón and Plaza, 2007). The $n-th$ Fibonacci numbers is the sum of the numbers of the two previous terms where $n \geq 2$ and the first and second terms are 0 and 1. In addition to Fibonacci numbers, there are Lucas and Mulatu numbers where the $n-th$ term with $n \geq 2$ is the sum of the numbers of the two previous terms, so that Lucas and Mulatu numbers have the same recursive function as Fibonacci numbers (Patel, D., and Lemma, 2011; Lemma et al., 2016; Lemma, 2019).

Lucas's numbers was found by Francois Edouard Anatole Lucas (Koshy, 2001). Lucas's numbers is obtained by taking the two initial terms, the 0-*th* term is 2 and the 1*st* term is 1. Mulatu numbers were discovered by Lemma Mulatu and published in 2011 (Mulatu, 2016). Mulatu numbers is obtained by taking two initial terms, the 0-*th* term is 4 and the 1*st* term is 1.

Interesting properties of Lucas's numbers have been reviewed by Lemma (2011). Schneider (2016) discusses the golden ratio in the form of continued fraction and nested radicals. Whereas Sivaraman (2020) developed Metallic ratio which is a generalization of three types of ratios namely Golden, Silver and Bronze ratios.

Similarities in the formation of Fibonacci numbers, Lucas and Mulatu numbers produce ideas to obtain new numbers which are generalizations of the three. The resulting generalization also produces other numbers that have not yet been found. Interesting properties of the new numbers are examined in this article and the results are obtained that these properties have been attached to Fibonacci, Lucas and Mulatu numbers.

2. Research Methodology

The purpose of this study is to find a sequence of numbers which is a generalization of rows of Fibonacci, Lucas and Mulatu numbers. Interesting properties of the new sequence numbers are examined in the article. The steps taken to achieve these two objectives are:

1. define a new sequence of numbers which is a generalization of Fibonacci Lucas and Mulatu numbers. This new sequence of numbers is called Generalization of Fibonacci-Lucas-Mulatu (GFLM) numbers;
2. shows that the solution of the GFLM numbers recursive relation is $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$;
3. build the Binet formula for GFLM numbers and show that the formula produces the Binet formula for Fibonacci, Lucas and Mulatu numbers;
4. shows that the ratio of two consecutive terms in the GFLM number converges to the golden ratio; establishing a silver ratio and a bronze ratio for the GFLM numbers;
5. build a Metallic ratio of order for GFLM numbers. This section also discusses several special cases related to Metallic ratio; and
6. states the Metallic ratio in the form of continued fractions and nested radicals.

3. Results and Discussion

3.1. The Fibonacci-Lucas-Mulatu (GFLM) Numbers

Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, … (Fernando and Prabowo, 2019). Lucas numbers are 2: 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, ….. (Koshy, 2001). Mulatu numbers are 4, 1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309, 500, 809, 1309, 2118, 3427, 5545, 8972, 14517, … (Mulatu, 2016). The three types of numbers have the same formation pattern, namely the recursive formula in Equation (1):

$$P_n = P_{n-1} + P_{n-2}\,;\, n \geq 2 \tag{1}$$

where

1. $P_0 = 0$ and $P_1 = 1$ for Fibonacci numbers;
2. $P_0 = 2$ and $P_1 = 1$ for Lucas numbers; and
3. $P_0 = 4$ and $P_1 = 1$ for Mulatu numbers.

The 2-*nd* term for the three types of numbers is 1. Whereas the first term can be stated with $2k$ where $k = 0,1,2.$ For $k = 0,1,2,3,4,\ldots..$obtained the Fibonacci-Lucas-Mulatu (GFLM) numbers:

$$2k, 1, 2k+1, 2k+2, 4k+3, 6k+5, 10k+8, 16k+13, 26k+21, \ldots.. \tag{2}$$

Definition 1. *The Fibonacci-Lucas-Mulatu (GFLM) numbers is a sequence of numbers obtained by a recursive formula* $P_n = P_{n-1} + P_{n-2}\,;\, n \geq 2$ *where* $P_0 = 2k$ *;* $k = 0,1,2,3,4,\ldots..$*and* $P_1 = 1$.

The GFLM numbers are extensions of Fibonacci, Lucas and Mulatu numbers. In other words, the three numbers are a special case of the GFLM numbers. Here are the first 16 terms for those obtained from Equation (2) or Theorem 1.

Table 1. Fibonacci, Lucas and Mulatu numbers are obtained from GFLM numbers

	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$k=1$	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364
$k=2$	4	1	5	6	11	17	28	45	73	118	191	309	500	809	1309	2118

One of the conveniences of recursive relations is that the equation in Definition 1 can be stated repeatedly:

$$\begin{aligned} P_n &= P_{n-1} + P_{n-2} \\ &= P_{n-1} + \left(P_{n-3} + P_{n-4}\right) \\ &= P_{n-1} + P_{n-3} + \left(P_{n-5} + P_{n-6}\right) \\ &= P_{n-1} + P_{n-3} + P_{n-5} + \left(P_{n-7} + P_{n-8}\right) \end{aligned}$$

and so on.

Note the GFLM numbers in Equation (2). The recursive equation for (2) is given in Definition 1, i.e. $P_n = P_{n-1} + P_{n-2}\,; n \geq 2$ where $P_0 = 2k\,;$ $k = 0,1,2,3,4,\ldots$ and $P_1 = 1$, we get

$$\begin{aligned} P_2 &= P_1 + P_0 \\ P_3 &= P_2 + P_1 \\ P_4 &= P_3 + P_2 = P_3 + P_1 + P_0 \\ P_5 &= P_4 + P_3 = P_4 + P_2 + P_1 \\ P_6 &= P_5 + P_4 = P_5 + P_3 + P_1 + P_0 \\ P_7 &= P_6 + P_5 = P_6 + P_4 + P_2 + P_1 \\ P_8 &= P_7 + P_6 = P_7 + P_5 + P_1 + P_1 + P_0 \end{aligned}$$

and so on.

Proposition 1. *If N is an odd number, then $P_N = P_1 + P_2 + P_4 + \ldots + P_{N-1}$ where $P_1 = 1$.*

Proposition 2. *If N is an even number, then $P_N = P_0 + P_1 + P_3 + P_5 + \ldots + P_{N-1}$ where $P_0 = 2k\,;$ $k = 0,1,2,3,4,\ldots$*

So, if N is an even number, then the $N-th$ GFLM numbers is the sum of odd terms of GFLM numbers fourth $(N-1)-th$ plus 0-*th* term. If N is an odd number, then the $N-th$ GFLM numbers is the sum of even terms of GFLM numbers fourth $(N-1)-th$ plus first term.

3.2. Recursive Relation

Homogeneous Linear Recursive Relations with Constant Coefficients (HLRRCC) is equation $f(n) = 0$ with all terms in a one-rank recursive relation, not multiplication of several terms, and the coefficient of all terms is a constant. HLRRCC with order $k-th$ given in Equation (3)

$$a_n = C_1 \cdot a_{n-1} + C_2 \cdot a_{n-2} + \ldots + C_k \cdot a_{n-k} \tag{3}$$

a_n : $n-th$ term; $C_1, C_2, \ldots, C_k$: constant and $C_k \neq 0$

The $n-th$ term of Fibonacci numbers, Lucas, and Mulatu numbers is the sum of the numbers of the two previous terms where $n \geq 1$. The three types of numbers are HLRRCC second order that satisfies Equation (4)

$$a_n = \alpha \cdot a_{n-1} + \beta \cdot a_{n-2} \tag{4}$$

a_n : $n-th$ term; α, β : nonzero real constant.

The solution of Equation (4) is sought by assuming $a_n = r^2$ so we get Equation (5)

$$r^2 - \alpha \cdot r - \beta = 0 \tag{5}$$

Real roots of quadratic equation (5) are expressed with $r_{1,2}$ that is $r_{1,2} = \dfrac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}$

The recursive relation of Equation (5) can be formed into the general solution of the recursive relation. Proof of a general solution of the recursive relation can be seen in Theorem 1.

Theorem 1. *Suppose that r_1 and r_2 the solution differs from the equation $r^2 - \alpha \cdot r - \beta = 0$, where $\alpha, \beta \in \Re$ and $\beta \neq 0$. Every solution from HLRRCC $a_n = \alpha \cdot a_{n-1} + \beta \cdot a_{n-2}$ has a solution from its recursive relation, i.e. $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$*

Proof: We will show that $a_n = \alpha \cdot a_{n-1} + \beta \cdot a_{n-2}$ has the solution of the recursive relation is $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$. Theorem 1 can be proved by means of an equation $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$ substituted into Equation (4).Note that,

$$\begin{aligned}
a_n &= \alpha \cdot a_{n-1} + \beta \cdot a_{n-2} \\
a_n &= \alpha \cdot \left[C_1 \cdot (r_1)^{n-1} + C_2 \cdot (r_2)^{n-1}\right] + \beta \cdot \left[C_1 \cdot (r_1)^{n-2} + C_2 \cdot (r_2)^{n-2}\right] \\
a_n &= C_1 \cdot (r_1)^{n-2}(\alpha \cdot r_1 + \beta) + C_2 \cdot (r_2)^{n-2}(\alpha \cdot r_2 + \beta) \\
a_n &= C_1 \cdot (r_1)^{n-2}(r_1)^2 + C_2 \cdot (r_2)^{n-2}(r_2)^2 \\
a_n &= C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n
\end{aligned}$$

So, it is proven that HLRRCC $a_n = \alpha \cdot a_{n-1} + \beta \cdot a_{n-2}$ have a solution of a recursive relation is $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$.

3.3. Fibonacci, Lucas, and Mulatu Numbers

Fibonacci, Lucas, and Mulatunumbers have the same equation i.e.:

$$a_n = a_{n-1} + a_{n-2} \tag{6}$$

Equation (6) can be changed to Equation (5) with real roots, i.e. r_1 and r_2 are

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2} \tag{7}$$

Fibonacci numbers have two initial terms, i.e. $F_0 = 0$ and $F_1 = 1$. Values C_1 and C_2 in $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$ on the Theorem 1 for Fibonacci numbers are expressed with $C_{1,F}$ and $C_{2,F}$. For $n = 0$ and $n = 1$, substitute on $a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$ obtained

$$C_{1,F} + C_{2,F} = 0 \tag{8}$$

$$C_{1,F}\left(\frac{1+\sqrt{5}}{2}\right) + C_{2,F}\left(\frac{1-\sqrt{5}}{2}\right) = 1 \tag{9}$$

With elimination method, the solution of Equations (8) and (9) are $C_{1,F} = \dfrac{1}{\sqrt{5}}$ and $C_{2,F} = -\dfrac{1}{\sqrt{5}}$. So, the equation of the Fibonacci number to $n-th$ term is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{10}$$

Lucas's number has two initial terms namely $L_0 = 2$ and $L_1 = 1$. In the same way, obtained $C_{1,L}$ = 1 and $C_{2,L}$ = 1. So, the equation of the Lucas number to $n-th$ term is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{11}$$

Mulatu numbers have two initial terms, i.e. $M_0 = 4$ and $M_1 = 1$. The coefficient of Mulatu numbers are $C_{1,M} = \frac{10-\sqrt{5}}{5}$ and $C_{2,M} = \frac{10+\sqrt{5}}{5}$. So, the equation of the Mulatu number to $n-th$ term is

$$M_n = \frac{10-\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{10+\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{12}$$

Write $\phi = \frac{1+\sqrt{5}}{2}$ and $1-\phi = \frac{1-\sqrt{5}}{2}$, then Equations (10), (11), and (12) respectively can be expressed with

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}(1-\varphi)^n \tag{13}$$

$$L_n = \phi^n + (1-\phi)^n \tag{14}$$

$$M_n = \frac{10-\sqrt{5}}{5}\phi^n + \frac{10+\sqrt{5}}{5}(1-\phi)^n \tag{15}$$

Equations (13), (14), and (15) respectively are Binet formulas for Fibonacci numbers, Lucas, and Mulatu. In general, Binet formula can be stated with

$$P_n = C_1 \cdot \phi^n + C_2 \cdot (1-\phi)^n \tag{16}$$

3.4. Golden Ratio on the GFLM Numbers

For the GFLM numbers (2): $2k, 1, 2k+1, 2k+2, 4k+3, 6k+5, 10k+8, 16k+13, \ldots\ldots$ with the first and second terms are $P_0 = 2k; k = 0,1,2,\ldots.$ and $P_1 = 1$. From Equation (16), for $n = 0$ and $n = 1$ obtained

$$\begin{aligned} C_1 \cdot \phi^0 + C_2 \cdot (1-\phi)^0 &= 2k \\ C_1 \cdot \phi^1 + C_2 \cdot (1-\phi)^1 &= 1 \end{aligned} \tag{17}$$

The solutions of Equation (17) are $C_1 = \frac{2k-1-2k\phi}{1-2\phi}$ and $C_2 = \frac{1-2k\phi}{1-2\phi}$. By taking $\phi = \frac{1+\sqrt{5}}{2}$, then $C_1 = \frac{k\sqrt{5}-k+1}{\sqrt{5}}$ and $C_2 = \frac{k\sqrt{5}+k-1}{\sqrt{5}}$; $k = 0,1,2,\ldots.$

1. For $k = 0$ obtained constant $C_1 = \frac{1}{\sqrt{5}} = C_{1,F}$ and $C_2 = -\frac{1}{\sqrt{5}} = C_{2,F}$
2. For $k = 1$ obtained constant $C_1 = 1 = C_{1,L}$ and $C_2 = 1 = C_{2,L}$
3. For $k = 2$ obtained constant $C_1 = \frac{10-\sqrt{5}}{5} = C_{1,M}$ and $C_2 = \frac{10+\sqrt{5}}{5} = C_{2,M}$

Referring to Equation (16), the $n-th$ term from the GFLM numbers on Equation (2) or Definition 1 can be stated by Equation (18) which is the Binet formula for the GFLM numbers

$$P_n = \left(\frac{k\sqrt{5}-k+1}{\sqrt{5}}\right)\cdot\phi^n + \left(\frac{k\sqrt{5}+k-1}{\sqrt{5}}\right)\cdot(1-\phi)^n$$
$$n = 0,1,2,3,\ldots;\ \forall k = 0,1,2,3,\ldots \tag{18}$$

Because $1-\phi = \frac{1-\sqrt{5}}{2} = -0.618$, then $-1 < 1-\phi < 1$. For $n \to \infty$, obtained $(1-\phi)^n \to 0$. This result is used to prove theorem 2.

Theorem 2. *The ratio of two consecutive terms in the GFLM number converges to the golden ratio, viz*

$$\frac{P_{n+1}}{P_n} = \frac{\left(\frac{k\sqrt{5}-k+1}{\sqrt{5}}\right)\cdot\phi^{n+1} + \left(\frac{k\sqrt{5}+k-1}{\sqrt{5}}\right)\cdot(1-\phi)^{n+1}}{\left(\frac{k\sqrt{5}-k+1}{\sqrt{5}}\right)\cdot\phi^{n} + \left(\frac{k\sqrt{5}+k-1}{\sqrt{5}}\right)\cdot(1-\phi)^{n}} = \frac{\phi^{n+1}}{\phi^n} = \phi$$

Define operator E with $E^r(P_n) = P_{n+r}$; $n \geq 0$ and $r \geq 0$. Note that $E^0(P_n) = P_n$, $E^1(P_n) = P_{n+1}$, and $E^2(P_n) = P_{n+2}$. Therefore, $P_{n+2} = E^2(P_n) = E^2$; $P_{n+1} = E^1(P_n) = E^1 = E$; and $P_n = E^0(P_n) = E^0 = 1$. From the GFLM numbers in Definition 1

$$\begin{aligned} P_n &= P_{n-1} + P_{n-2} \\ P_{n+2} &= P_{n+1} + P_n \\ P_{n+2} - P_{n+1} - P_n &= 0 \\ E^2(P_n) - E^1(P_n) - E^0(P_n) &= 0 \\ \left(E^2 - E^1 - E^0\right)(P_n) &= 0 \\ \left(E^2 - E - 1\right)(P_n) &= 0 \end{aligned}$$

The last equation $\left(E^2 - E - 1\right)(P_n) = 0$ is similar to $m^2 - m - 1 = 0$ with a solution $m = \frac{1 \pm \sqrt{5}}{2}$. One solution is called the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. Another solution is $1-\phi = \frac{1-\sqrt{5}}{2}$. As a result, the GFLM number in Definition 1 is $P_n = P_{n-1} + P_{n-2}$ can be written as Eq. (19)

$$P_n = C_1 \cdot \phi^n + C_2 \cdot (1-\phi)^n \ ; n \geq 0 \tag{19}$$

In the next section we will build a new line that is raised from the GFLM number. From these new numbers, two ratios are called the Silver ratio and the Bronze ratio. In the next section, a Metallic ratio is built which is a generalization of the ratio of gold, silver and bronze.

3.5. Silver Ratio on the GFLM Numbers

Consider at the sequence of numbers in (20):

$$2k, 1, 2k+2, 4k+5, 10k+12, 24k+29, 58k+70, 140k+169, \ldots\ldots \tag{20}$$

For $k = 0,1,2$ obtained the numbers in Table 2. The ratio of two consecutive numbers for Table 2 converges to 2.4142.

Table 2. Fibonacci numbers, Lucas and Mulatu Order 2

$k=0$	0	1	2	5	12	29	70	169	408	985	2378	5741	13.860
$k=1$	2	1	4	9	22	53	128	309	746	1801	4348	10.497	25.342
$k=2$	4	1	6	13	32	77	186	449	1084	2617	6318	15.253	36.824

Rows of numbers (20) can be stated recursively with Equation (21):

$$P_n = 2P_{n-1} + P_{n-2}\,;\, n \geq 2 \tag{21}$$

where $P_0 = 2k\,;\, k = 0,1,2,3,\ldots.$ and $P_1 = 1$

With operator E, Equation (21) is equivalent to $(E^2 - 2E - 1)(P_n) = 0$ which is also equivalent to $m^2 - 2m - 1 = 0$. The solution of the last equation is $m = 1 \pm \sqrt{2}$ and solution $m = 1 + \sqrt{2}$ called the silver ratio, symbolized $\lambda = 1 + \sqrt{2} = 2.4142$. Consider that another solution is $2 - \lambda = 1 - \sqrt{2}$. As a result, $P_n = 2P_{n-1} + P_{n-2}$ it can be written as Eq. (22)

$$P_n = C_1 \cdot \lambda^n + C_2 \cdot (2-\lambda)^n\,;\, n \geq 0 \tag{22}$$

Limit the ratio of two consecutive numbers in (20) converging to silver ratio. Because $-1 < 2 - \lambda < 1$, then $(2-\lambda)^n \to 0$ for $n \to \infty$. Obtained ratio

$$\frac{P_{n+1}}{P_n} = \frac{C_1 \cdot \lambda^{n+1} + C_2 \cdot (2-\lambda)^{n+1}}{C_1 \cdot \lambda^n + C_2 \cdot (2-\lambda)^n} = \frac{C_1 \cdot \lambda^{n+1} + C_2 \cdot 0}{C_1 \cdot \lambda^n + C_2 \cdot 0} = \lambda$$

3.6. Bronze Ratio on the GFLM Numbers

Consider at the sequence of numbers in (23):

$$2k, 1, 2k+3, 6k+10, 20k+33, 66k+109, 152k+251, \ldots\ldots \tag{23}$$

For $k = 0,1,2$ obtained by the numbers in Table 3. The ratio of two consecutive numbers to $k = 0,1,2$ in Table 3 converge towards 3.3028.

Table 3. Fibonacci numbers, Lucas and Mulatu Order 3

$k=0$	0	1	3	10	33	109	360	1189	3927	12.970	42.837	141.481	467.280
$k=1$	2	1	5	16	53	175	578	1909	6305	20.824	68.777	227.155	750.242
$k=2$	4	1	7	22	73	241	796	2629	8683	28.678	94.717	312.829	1.033.204

Rows of numbers (23) can be stated recursively with Equation (24):

$$P_n = 3P_{n-1} + P_{n-2}\,;\, n \geq 2 \tag{24}$$

where $P_0 = 2k\,;\, k = 0,1,2,3,\ldots.$ and $P_1 = 1$

With operator E, Equation (24) equivalent to $(E^2 - 3E - 1)(P_n) = 0$ which is also equivalent to $m^2 - 3m - 1 = 0$. The solution of the last equation is $m = \frac{3 \pm \sqrt{13}}{2}$ and solution $m = \frac{3 + \sqrt{13}}{2}$ called the bronze ratio, symbollized $\mu = \frac{3+\sqrt{13}}{2} = 3.3028$. Consider that another solution is $3 - \mu = \frac{3 - \sqrt{13}}{2}$. As a result, $P_n = 3P_{n-1} + P_{n-2}$ can be written as Eq. (25)

$$P_n = C_1 \cdot \mu^n + C_2 \cdot (3-\mu)^n\,;\, n \geq 0 \tag{25}$$

Limit the ratio of two consecutive numbers at (25) converging to Bronze ratio. Because $-1 < 3 - \mu < 1$ then $(3-\mu)^n \to 0$ for $n \to \infty$. Obtained ratio

$$\frac{P_{n+1}}{P_n}=\frac{C_1\cdot\mu^{n+1}+C_2\cdot(3-\mu)^{n+1}}{C_1\cdot\mu^{n}+C_2\cdot(3-\varpi)^{n}}=\frac{C_1\cdot\mu^{n+1}+C_2\cdot 0}{C_1\cdot\mu^{n}+C_2\cdot 0}=\mu$$

3.7. Metallic Ratio on the GFLM Numbers

We have obtained three types of ratios namely the ratio of Gold, Silver and Bronze with successive values is $\phi=1.618$; $\lambda=2.4142$; and $\mu=3.3028$. The three types of ratios are special cases of Metallic Ratio.

Define recurrence relations in Equation (26)

$$P_n=t\cdot P_{n-1}+P_{n-2}\quad ;\ n\geq 2,\ \ t=1,2,3,\ldots \tag{26}$$

where $P_0=2k\,;k=0,1,2,3,\ldots$ and $P_1=1$

With operator E, we get $\left(E^2-tE-1\right)P_n=0$. The auxiliary equation is $m^2-tm-1=0$ which is a quadratic equation with a solution $m_{1,2}=\frac{t\pm\sqrt{t^2+4}}{2}$. Solution $m=\frac{t+\sqrt{t^2+4}}{2}$ called Metallic ratio of order t and symbolized with $\rho_t=\frac{t+\sqrt{t^2+4}}{2}$, $t=1,2,3,\ldots$. Another root is written $t-\rho_t=\frac{t-\sqrt{t^2+4}}{2}$.

3.7.1. Special Case

Metallic ratios of order 1, 2 and 3, respectively, are Golden, Silver and Bronze ratios. The name follows the medal on the Olimpycs.

1. If $t=1$, then we get $\rho_1=\frac{1+\sqrt{5}}{2}=\phi$, the Golden ratio
2. If $t=2$, then we get $\rho_2=\frac{2+\sqrt{8}}{2}=1+\sqrt{2}=\lambda$, the Silver ratio
3. If $t=3$, then we get $\rho_3=\frac{3+\sqrt{13}}{2}=\mu$, the Bronze ratio

3.7.2. The Infinite Small Order Converge to Zero

For any real number t which is very small, we can write $t=\frac{1}{q^r}$, where q is large and $r>0$. In this case, the Metallic ratio by order t will take the form $\rho_t=\frac{1+\sqrt{1+4q^{2r}}}{2q^r}$. For $q\to\infty$ so that $t\to 0$ and $\rho_t=\frac{1+\sqrt{1+4q^{2r}}}{2q^r}=\frac{1}{2}\cdot\frac{1}{q^r}+\frac{1}{2}\sqrt{4+\frac{1}{q^{2r}}}\to\frac{1}{2}\cdot 0+\frac{1}{2}\sqrt{4+0}=1$. Thus, Metallic ratio approach 1 when t is very small with order $t=q^{-r}$

For $r=1$ so that $t=\frac{1}{q}$ and $\rho_{\frac{1}{q}}=\frac{1+\sqrt{1+4q^2}}{2q}$ $q=1,2,3,\ldots$ called Platinum ratio. For $r=2$ then $\rho_{\frac{1}{q^2}}=\frac{1+\sqrt{1+4q^4}}{2q^2}$ $q=1,2,3,\ldots$ called Rhodium ratio.

For example $t_1=\frac{1}{q^{r_1}}$ and $t_2=\frac{1}{q^{r_2}}$. For $r_1>r_2$ then $q^{r_1}>q^{r_2}$ and $t_1>t_2$. That is, the bigger r then $t=\frac{1}{q^r}$ faster towards 0. Accordingly, for q the same, Metallic ratio $\rho_t=\frac{1+\sqrt{1+4q^{2r}}}{2q^r}$ will be faster towards 1 for r the greater one. So, the Rhodium ratio is faster to 1 than the platinum ratio.

Now we review for $r=1$ which produce Platinum ratio $\rho_{\frac{1}{q}}=\frac{1+\sqrt{1+4q^2}}{2q}$ for $q=1,2,3,\ldots$.Take $q=10$ and $q=100$ Obtained $\rho_{\frac{1}{10}}=\frac{1+\sqrt{1+400}}{20}=1.05124922$ and $\rho_{\frac{1}{100}}=\frac{1+\sqrt{1+40{,}000}}{200}=1.0050125$. Thus, Platinum ratio faster towards 1 for q bigger ones.

For $r=2$, Rhodium ratio $\rho_{\frac{1}{q}}=\frac{1+\sqrt{1+4q^4}}{2q^2}$ For $q=10$ obtained $\rho_{\frac{1}{100}}=\frac{1+\sqrt{1+40{,}000}}{200}=1.0050125$. For this case, we get the result that the Platinum ratio and Rhodium ratio are the same.

3.7.3. The Infinite Big Order

For t large, Metallic ratio will increase. We get, for $t\to\infty$

$$\rho_t=\frac{t+\sqrt{t^2+4}}{2}\approx\frac{t+\sqrt{t^2}}{2}=\frac{2t}{2}=t$$

Thus, for $t\to 0$ then $\rho_t\to 1$ and for $t\to\infty$ then $\rho_t\to t$

3.7.4. The Ratio of the Metallic Ratio

Now we will compare the ratio of the Metallic ratio with successive orders:

$$\frac{\rho_{t+1}}{\rho_t}=\frac{\frac{(t+1)+\sqrt{(t+1)^2+4}}{2}}{\frac{t+\sqrt{t^2+4}}{2}}=\frac{(t+1)+\sqrt{(t+1)^2+4}}{t+\sqrt{t^2+4}}=1\,;t\to\infty$$

That is, for t larger, the k *-th* Metallic ratio is almost identical to ($k+1$)*-th* Metallic ratio.

3.8. Expressing Metallic Ratio in Continued Fractions

Consider the Equation $m^2-tm-1=0$ with one solution is the Metallic ratio, ie $\rho_t=\frac{t+\sqrt{t^2+4}}{2}$. Consequence of $m^2-tm-1=0$ obtained $\rho_t^2-t\cdot\rho_t-1=0$ or $\rho_t^2=1+t\cdot\rho_t$. If the two segments are divided by ρ_t obtained $\rho_t=1+\frac{1}{\rho_t}$

Proposition 3. *The Metallic ratio is equal to its own reciprocal plus 1:*

$$\rho_t=1+\frac{1}{\rho_t},\ t=1,2,3,\ldots.$$

We can express Metallic ratio in the form of continued fraction. Note that

$$\rho_t = 1+\frac{1}{\rho_t} = 1+\cfrac{1}{1+\cfrac{1}{\rho_t}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\rho_t}}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\rho_t}}}} = \ldots\ldots$$

Proposition 4. *We can write the Metallic ratio as an infinite continued fraction with all coefficients equal to 1 except the last coefficinet:*

$$\rho_t = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots\ldots}}}},\ t=1,2,3,\ldots.$$

3.9. State the Metallic Ratio in the Nested Radical Form

Consider the Equation $m^2 - tm - 1 = 0$ with one solution is the Metallic ratio ie $\rho_t = \frac{t+\sqrt{t^2+4}}{2}$ Consequence of $m^2 - tm - 1 = 0$ obtained $\rho_t^2 - t\cdot\rho_t - 1 = 0$ or $\rho_t^2 = 1 + t\cdot\rho_t$. If both segments are drawn, the roots are obtained $\rho_t = \sqrt{1+t\cdot\rho_t}$

Proposition 5. *The Metallic ratio is equal to the square of itself plus 1:*

$$\rho_t = \sqrt{1+t\cdot\rho_t},\ t=1,2,3,\ldots.$$

We can express Metallic ratio in the form of nested radical. Note that for $t=1,2,3,\ldots.$ applies:

$$\rho_t = \sqrt{1+t\cdot\rho_t}$$
$$= \sqrt{1+t\cdot\sqrt{1+t\cdot\rho_t}} = \sqrt{1+t\cdot\sqrt{1+t\cdot\sqrt{1+t\cdot\rho_t}}} = \sqrt{1+t\cdot\sqrt{1+t\cdot\sqrt{1+t\cdot\sqrt{1+t\cdot\rho_t}}}} = \ldots.$$

Proposition 6. *The Metallic ratio is equal to*

$$\rho_t = \frac{t+\sqrt{t^2+4}}{2} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\rho_t}}}} = \sqrt{1+t\cdot\sqrt{1+t\cdot\sqrt{1+t\cdot\sqrt{1+t\cdot\ldots\ldots}}}}$$

4. Conclussion

In this article, we generalize to three types of numbers namely Fibonacci numbers, Lucas numbers and Mulatu numbers. The result is a new number that we call the GFLM numbers. The interesting properties of the GFLM we describe in this article include the Binet formula for the GFLM number, and the golden ratio, the silver ratio and the Bronze ratio of the GFLM number. We also present generalizations of these three types of ratios, called Metallic ratios. In the last part we state the Metallic ratio in the form of continued fraction and nested radicals.

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